After a brief introduction and review of the basic notions in stochastic processes, the stochastic quantization method, proposed by G. Parisi and Y.-S. Wu in 1981, and its applications and (hopefully) recent developments are explained.

The contents of the lecture will be

1. Introduction
2. Basics of stochastic quantization of Parisi and Wu
3. Some of the achievements in early days
4. Recent developments

The lecture will be based mainly on the following review articles

- M. Namiki, Stochastic Quantization, Lecture Notes in Physics, Springer-Verlag (1992)

See also the original paper
Chapter 1. INTRODUCTION

Essence of stochastic quantization of Parisi–Wu

**Euclidean Field Theory = equilibrium limit of a (fictitious) stochastic process**

or stated differently

\[(d + 1)\text{-dimensional (classical) stochastic process} \]
\[\text{reduces to} \]
\[d\text{-dimensional (quantized) field theory in equilibrium} \]

... dimensional reduction

diamond background idea: quantum fluctuation ⇔ “classical” stochastic process (??)

Remarks

- introduction of 5th (in 4 dimensions), or \((d + 1)\)th “time,” originally called fictitious time or computer time, over which a (hypothetical) stochastic process is assumed
- presence of a hypothetical thermal reservoir assumed (⇔ origin of quantum fluctuation)
- original proposal by Parisi–Wu was more practical-use orientated, i.e., quantization of gauge field without fixing a gauge!
- there is another stochastic quantization method based on the real time stochastic process
  - Nelson’s stochastic quantization
- also another quantization method closely related to this quantization method
  - micro-canonical quantization ((\(d + 1\))-dimensional classical theory \(\rightarrow\) \(d\)-dimensional quantum theory)

... these two methods will not be treated here
1.1 Basic concepts of stochastic processes

**Stochastic variables** ... representing random thermal fluctuations from heat reservoir

- Brownian motion, fluid dynamics,...

○ basic concepts for stochastic processes, like *Langevin equation, Fokker–Planck equation, Markovian processes*

... developed at the beginning of 20th century

**Quick look at basic elements in stochastic processes**

★ **stochastic variable** \( X \) = set of possible values \( x, \{x\} \), ranging \( x \in I \), endowed with a probability density \( P(x) \)

\[
P(x) \geq 0 \text{ (positive semi-definite)}, \quad \int_I P(x)dx = 1 \text{ (normalizable)}
\]

- function of stochastic variable \( f(X) \): again a stochastic variable

★ **stochastic process** \( Y(t) \equiv f(t, X) \) = function of time \( t \) and stochastic variable \( X \)

- expectation value of \( Y(t) \)

\[
\langle Y(t) \rangle = \int_I Y_x(t)P(x)dx, \quad \text{where} \quad Y_x(t) = f(t, x) \quad : \text{sample function or a realization of stochastic process}
\]

★ **joint probability**

\[
P_n(y_1, t_1; y_2, t_2; \ldots; y_n, t_n) = \int_I \delta(y_1 - Y_x(t_1))\delta(y_2 - Y_x(t_2)) \ldots \delta(y_n - Y_x(t_n))P(x)dx
\]

- clearly

\[
\int P_n(y_1, t_1; y_2, t_2; \ldots; y_n, t_n)dy_1 = P_{n-1}(y_2, t_2; \ldots; y_n, t_n) \quad \text{etc.}
\]

★ **conditional probability** = probability conditioned

\[
P_{\ell/k}(y_{k+1}, t_{k+1}; \ldots; y_{k+\ell}, t_{k+\ell}/y_1, t_1; \ldots; y_k, t_k) = P_{k+\ell}(y_1, t_1; \ldots; y_{k+\ell}, t_{k+\ell})/P_k(y_1, t_1; \ldots; y_k, t_k)
\]
**Markov process** = conditional probability depends only on the last variable

introduce an ordering \( t_1 < t_2 < \cdots < t_n \), then for Markov process,

\[
P_{1/n-1}(y_n, t_n/ y_1, t_1; \ldots; y_{n-1}, t_{n-1}) \equiv P_{1/1}(y_n, t_n/ y_{n-1}, t_{n-1}) = P_2(y_{n-1}, t_{n-1}; y_n, t_n)/P_1(y_{n-1}, t_{n-1})
\]

therefore

\[
P_2(y_{n-1}, t_{n-1}; y_n, t_n) = P_{1/1}(y_n, t_n/ y_{n-1}, t_{n-1})P_1(y_{n-1}, t_{n-1})
\]

also a Chapman-Kolmogorov equation follows

\[
P_1(y_n, t_n) = \int P_2(y_{n-1}, t_{n-1}; y_n, t_n)dy_{n-1} = \int P_{1/1}(y_n, t_n/ y_{n-1}, t_{n-1})P_1(y_{n-1}, t_{n-1})dy_{n-1}
\]

♢ typical example: Brownian motion

- a particle with a large mass \( m \) immersed in a fluid
dynamics is phenomenologically described by the **Langevin equation**

\[
m\ddot{v}(t) = -\alpha v(t) + \eta(t)
\]

where \( \begin{cases} v(t) & \text{velocity} \\ \alpha > 0 & \text{friction constant} \\ \eta(t) & \text{stochastic force (noise)} \end{cases} \)

phenomenologically representing collisions with molecules of the fluid

velocity \( v(t) \) should be a Markov process, for the number of collisions depends on the present value of velocity, not before

\( \leftrightarrow \) stochastic force \( \eta(t) \) is also a Markov process

▷ Gaussian probability distribution (functional) reads

\[
P(\eta) = \frac{e^{-\frac{1}{4D} \int \eta^2(t) dt}}{\int \mathcal{D}\eta e^{-\frac{1}{4D} \int \eta^2(t) dt}} = \mathcal{N}^{-1}e^{-\frac{1}{4D} \int \eta^2(t) dt}, \quad \int \mathcal{D}\eta P(\eta) = 1, \quad D > 0 : \text{diffusion constant, to be determined}
\]

introduce a generating functional for noise correlations

\[
\mathcal{P}(J) = \langle e^{\int J(t) \cdot \eta(t) dt} \rangle = \mathcal{N}^{-1} \int \mathcal{D}\eta e^{-\frac{1}{4D} \int \eta^2(t) dt + \int J(t) \cdot \eta(t) dt} = e^D \int J^2(t) dt
\]
then

\[
\langle \eta_i(t) \rangle = \left. \frac{\delta}{\delta J_i(t)} \mathcal{P}[J] \right|_{J=0} = 2D J_i(t) \mathcal{P}[J] \bigg|_{J=0} = 0
\]

\[
\langle \eta_i(t) \eta_k(t') \rangle = \left. \frac{\delta^2}{\delta J_i(t) \delta J_k(t')} \mathcal{P}[J] \right|_{J=0} = 2D \delta_{ik} \delta(t - t')
\]

in general,

\[
\langle \eta_i(t_1) \cdots \eta_i(t_n) \rangle = \begin{cases} 
0 & \text{for odd } n \\
\sum \text{all comb. } k, \ell \langle \eta_i(t_k) \eta_i(t_\ell) \rangle & \text{for even } n
\end{cases}
\]

thus, the stochastic force (noise) \( \eta \) is characterized by

*Gaussian* ... every noise correlation breaks into products of two-point correlations
*white noise* ... spectrum of two-point function is flat in Fourier space (*white*)

Notice \( \delta \)-function in noise correlation is singular \( \longrightarrow \) *Wiener process*: mathematically well-defined object
- introduce formally

\[
\text{Wiener process: } W(t) = \int_0^t \eta(t') dt' \quad \text{or} \quad dW(t) = \eta(t) dt
\]

- correlations (of course, Gaussian)

\[
\langle W(t) \rangle = 0
\]

\[
\langle W_i(t) W_j(t') \rangle = \int_0^t dt_1 \int_0^{t'} dt_2 \langle \eta_i(t) \eta_j(t') \rangle = 2D \int_0^t dt_1 \int_0^{t'} dt_2 \delta_{ij} \delta(t_1 - t_2) = 2D \delta_{ij} \min(t, t')
\]

- the Langevin equation in the integrated form

\[
mdv(t) = -\alpha v(t) dt + dW(t) \quad \text{or} \quad mv(t) - mv(t_0) = -\alpha \int_{t_0}^t v(\tau) d\tau + \int_{t_0}^t dW
\]

- the last term (integral over stochastic variable) needs proper definition! (*stochastic calculus*)
★ stochastic calculi: Ito and Stratonovich

- consider an integral over a stochastic variable \( W \)

\[
\int_{t_0}^{t} G(W) dW \equiv \lim_{n \to \infty} S_n, \quad S_n = \sum_{i=1}^{n} G(\tau_i)(W(t_i) - W(t_{i-1})), \quad t_0 \leq t_1 \leq \cdots \leq t_n = t, \quad \tau_i \in [t_{i-1}, t_i],
\]
\[
\forall G(\tau_i) : \text{a function(al) dependent on } \tau_i \text{ and } W(\tau) \text{ with } \tau \leq \tau_i
\]

\>

choice of \( \tau_i \) (i.e., \( a \) in the following) \textbf{does matter} the result!

\[
\tau_i = (1-a)t_{i-1} + at_i, \quad 0 \leq a \leq 1
\]

Stochastic calculus is prescription-dependent!

... due to anomalous behavior of stochastic variable, i.e., continuous but \textbf{not differentiable}!

- two of the most famous stochastic calculi

  \textit{Ito calculus}: \( a = 0 \)

\[
S_n^{\text{Ito}} = \sum_{i=1}^{n} G(t_{i-1})(W(t_i) - W(t_{i-1}))
\]

  \textit{Stratonovich calculus}: \( a = 1/2 \) (mid-point prescription)

\[
S_n^{\text{Str}} = \sum_{i=1}^{n} G\left(\frac{t_i + t_{i-1}}{2}\right)(W(t_i) - W(t_{i-1}))
\]

In a mathematically rigorous sense, Ito calculus is preferred and important relation

\[
(dW(t))^2 = 2Ddt
\]

* Here a single component is assumed for \( W \) for simplicity.
holds in the sense of probability, i.e., ∀\(G\)

\[
\langle \int G(W)(dW)^2 \rangle = \langle \int 2DG(W)dt \rangle
\]

Actually, since \(G(t_{i-1})\) is statistically independent of \(W(t_i) - W(t_{i-1})\), i.e.,

\[
\langle G(t_{i-1})(W(t_i) - W(t_{i-1})) \rangle = \langle G(t_{i-1}) \rangle \langle (W(t_i) - W(t_{i-1})) \rangle = 0,
\]

because \(G(t_{i-1})\) is a function of stochastic variables up to time \(t_{i-1}\), we have

\[
\langle \sum_{i=1}^{n} G(t_{i-1})(W(t_i) - W(t_{i-1}))^2 \rangle = \sum_{i=1}^{n} \langle G(t_{i-1}) \rangle \langle (W(t_i) - W(t_{i-1}))^2 \rangle = \sum_{i=1}^{n} \langle G(t_{i-1}) \rangle 2D(t_i + t_{i-1} - 2t_{i-1})
\]

\[
= \sum_{i=1}^{n} \langle 2DG(t_{i-1})(t_i - t_{i-1}) \rangle.
\]

Similarly,

\[
(dW(t))^2 + n = 0, \quad n = 1, 2, \ldots
\]

\[
\Rightarrow dW(t) = O(\sqrt{dt}) \quad \text{... expansion up to } (dW)^2 \text{ is necessary!}
\]

⋆ Ito calculus: \(G(t)\) is independent of \(dW(t)\), but we have to keep order-(\(dW\))^2 terms

⋆ Stratonovich calculus (= mid-point prescription): close to the usual differential treatment

- In order to distinguish them, a symbol \(\circ\) can be used to represent a product in the Stratonovich sense

\[
G(W) \circ dW \leftarrow G\left(\frac{t_i + t_{i-1}}{2}\right)(W(t_i) - W(t_{i-1}))
\]

\[
\sim G\left(\frac{W(t_i) + W(t_{i-1})}{2}\right)(W(t_i) - W(t_{i-1}))
\]

\[
\sim G(t_{i-1})(W(t_i) - W(t_{i-1})) + \frac{1}{2}(G(t_i) - G(t_{i-1}))(W(t_i) - W(t_{i-1}))
\]
Formally,
\[ G(W) \circ dW = G(W)dW + \frac{1}{2}dG(W)dW \]

○ (Formal) solution of the Brownian motion

\[ v(t) = e^{-\frac{\alpha}{m}t}v(0) + \frac{1}{m} \int_0^t e^{-\frac{\alpha}{m}(t-t')}\eta(t')dt' \]

- average energy of the Brownian particle

\[ \langle E(t) \rangle = \frac{m}{2} \langle v^2(t) \rangle \]
\[ = e^{-2\frac{\alpha}{m}t}E(0) + \frac{1}{2m} \int_0^t dt_1 \int_0^t dt_2 e^{-\frac{\alpha}{m}(t-t_1)-\frac{\alpha}{m}(t-t_2)} \langle \eta(t_1) \cdot \eta(t_2) \rangle = e^{-2\frac{\alpha}{m}t}E(0) + \frac{3D}{2\alpha} \left( 1 - e^{-2\frac{\alpha}{m}t} \right) \]
\[ \rightarrow \frac{3D}{2\alpha} \quad \text{as} \quad t \rightarrow \infty, \]

which should be equal to \( \frac{3}{2}k_BT \) (\( k_B \): Boltzmann constant, \( T \): temperature of the fluid).

\[ \text{Fluctuation-dissipation theorem:} \quad D = \alpha k_BT \]

* Fokker–Planck equation: complementary (to the Langevin equation) view of stochastic process

Langevin equation: evolution of stochastic variable ↔ Fokker–Planck equation: evolution of probability distribution

cf. Heisenberg picture v.s. Schrödinger picture

- derivation of Fokker–Planck equation: (I) (essentially) Stratonovich calculus*

Introduce a conditional probability distribution function

\[ P(v, t) \equiv P_{1/1}(v, t/v_0, t_0), \quad \int dvP(v, t) = 1 \]

* But the symbol \( \circ \) suppressed for simplicity
∀ \( f(v) \),
\[
\langle f(v) \rangle_t = \langle f(v(t)) \rangle = \mathcal{N}^{-1} \int \mathcal{D}\eta e^{-\frac{1}{2\mathcal{N}} \int \eta^2(\tau) d\tau} f(v(t))
\]

: here the solution of the Langevin equation \( v(t) \) is given as a functional of \( \eta \)
\[
= \int dv f(v) P(v, t) \quad \text{... Fokker–Planck view point}
\]

Then,
\[
\frac{d}{dt} \langle f(v) \rangle_t = \mathcal{N}^{-1} \int \mathcal{D}\eta e^{-\frac{1}{2\mathcal{N}} \int \eta^2(\tau) d\tau} \dot{v}(t) \cdot \frac{\partial}{\partial v(t)} f(v(t))
\]
\[
= \mathcal{N}^{-1} \int \mathcal{D}\eta e^{-\frac{1}{2\mathcal{N}} \int \eta^2(\tau) d\tau} \left( -\frac{\alpha}{m} v(t) + \frac{1}{m} \eta(t) \right) \cdot \frac{\partial}{\partial v(t)} f(v(t)) \quad \leftarrow \text{Langevin equation } m\dot{v} = -\alpha v + \eta
\]
\[
= \left\langle \left( -\frac{\alpha}{m} v \right) \cdot \frac{\partial f(v)}{\partial v} \right\rangle_t + \mathcal{N}^{-1} \int \mathcal{D}\eta e^{-\frac{1}{2\mathcal{N}} \int \eta^2(\tau) d\tau} 2D \frac{\delta}{\delta \eta(t)} \cdot \frac{\partial}{\partial v(t)} f(v(t))
\]

As for the last term,
\[
\frac{\delta}{\delta \eta(t)} \cdot \frac{\partial}{\partial v(t)} f(v(t)) = \delta v_j(t) \frac{\partial^2 f(v(t))}{\delta \eta_i(t) \partial v_i(t) \partial v_j(t)} = \frac{1}{m} \int_0^t \delta_{ij} \delta(t-t') dt' \frac{\partial^2 f(v(t))}{\partial v_i(t) \partial v_j(t)}
\]
\[
= \theta(0) \frac{1}{m} \frac{\partial}{\partial v(t)} \cdot \frac{\partial f(v(t))}{\partial v(t)} = \frac{1}{2} \frac{1}{m} \frac{\partial}{\partial v(t)} \cdot \frac{\partial f(v(t))}{\partial v(t)}
\]

\[
\text{... mid-point (Stratonovich) prescription}
\]

Therefore,
\[
\frac{d}{dt} \langle f(v) \rangle_t = \left\langle \left( -\frac{\alpha}{m} v \right) \cdot \frac{\partial f(v)}{\partial v} \right\rangle_t + \frac{D}{m^2} \left( \frac{\partial}{\partial v} \cdot \frac{\partial f(v)}{\partial v} \right)_t
\]

This means, in terms of the probability distribution,
\[
\int dv f(v) \frac{\partial}{\partial t} P(v, t) = \int dv P(v, t) \left( -\frac{\alpha}{m} v \cdot \frac{\partial f(v)}{\partial v} + \frac{D}{m^2} \frac{\partial}{\partial v} \cdot \frac{\partial f(v)}{\partial v} \right) = \int dv f(v) \frac{\partial}{\partial v} \left( \frac{\alpha}{m} v + \frac{D}{m^2} \frac{\partial}{\partial v} \right) P(v, t)
\]
\[ \frac{\partial}{\partial t} P(v, t) = \frac{\partial}{\partial v} \left( \frac{D}{m^2} \frac{\partial}{\partial v} + \frac{\alpha}{m} v \right) P(v, t) \quad \text{Fokker–Planck equation} \]

- derivation of Fokker–Planck equation: (II) Ito calculus

The Langevin equation

\[ dv(t) = v(t + dt) - v(t) = -\frac{\alpha}{m} v(t) dt + \frac{1}{m} dW(t) \]

\[ \forall f, \quad df(v(t)) = f(v(t + dt)) - f(v(t)) = f(v(t) + dv(t)) - f(v(t)) = dv(t) \cdot \frac{\partial f(v(t))}{\partial v(t)} + \frac{1}{2!} dv_i(t) dv_j(t) \frac{\partial^2 f(v(t))}{\partial v_i(t) \partial v_j(t)} + O(\sqrt{dt}) \]

\[ = \left( -\frac{\alpha}{m} v(t) dt + \frac{1}{m} dW(t) \right) \cdot \frac{\partial f(v(t))}{\partial v(t)} + \frac{1}{2!} \frac{1}{m^2} dW_i(t) dW_j(t) \frac{\partial^2 f(v(t))}{\partial v_i(t) \partial v_j(t)} + O(\sqrt{dt}) \]

therefore

\[ d\langle f(v(t)) \rangle = -\frac{\alpha}{m} \langle v(t) \cdot \frac{\partial f(v(t))}{\partial v(t)} \rangle dt + \frac{1}{m} \langle dW(t) \cdot \frac{\partial f(v(t))}{\partial v(t)} \rangle + \frac{1}{2!} \frac{1}{m^2} \langle dW_i(t) dW_j(t) \frac{\partial^2 f(v(t))}{\partial v_i(t) \partial v_j(t)} \rangle + O(\sqrt{dt}) \]

\[ = -\frac{\alpha}{m} \langle v(t) \cdot \frac{\partial f(v(t))}{\partial v(t)} \rangle dt + \frac{1}{2!} \frac{1}{m^2} 2D \delta_{ij} dt \langle \frac{\partial^2 f(v(t))}{\partial v_i(t) \partial v_j(t)} \rangle + O(\sqrt{dt}) \]

\[ = \left( -\frac{\alpha}{m} v \right) \cdot \frac{\partial f(v)}{\partial v} \right)_t dt + D \left( \frac{\partial}{\partial v} \cdot \frac{\partial f(v)}{\partial v} \right)_t dt \quad \rightarrow \quad \text{same Fokker–Planck equation as above!} \]

- Equilibrium or stationary solution

\[ P_{eq}(v) = P_{st}(v) \propto e^{-\frac{1}{2} D m v^2} = e^{-\frac{\beta}{2} m v^2} : \text{Boltzmann distribution (}\beta = 1/k_B T\) \]

- needs more careful treatment for derivation of equilibrium distribution (see later)

- simple derivation of the 1-d. Langevin equation
Consider a 1-d. Brownian particle moving right: mass $M$, velocity $v(t) > 0$

- elastic collisions with fluid molecules of mass $m \ll M$
  from the left ($u_- > v$ assumed)

$$Mv + mu_- = Mv' + mu'_-, \quad v' - u'_- = -(v - u_-), \quad M \Delta v = -\frac{2Mm}{M+m}(v - u_-)$$

from the right

$$Mv - mu_+ = Mv' + mu'_+, \quad v' - u'_+ = -(v + u_+), \quad M \Delta v = -\frac{2Mm}{M+m}(v + u_+)$$

In total, the average force exerted on the particle

$$\bar{f} = -\frac{2Mm}{M+m}(v - u_-)(u_- - v) \rho - \frac{2Mm}{M+m}(v + u_+)(v + u_+) \rho = -\frac{2Mm}{M+m}(u_+ + u_-)(2v + u_+ - u_-) \rho$$

$$\sim -8\rho m \bar{u} v - 4\rho m \bar{u}(u_+ - u_-)$$

with $\rho$ being the molecular density and $\langle u_\pm \rangle = \bar{u}$. Therefore, we may identify

friction constant: $\alpha = 8\rho m \bar{u}$

random noise: $\eta = -4\rho m \bar{u}(u_+ - u_-)$

or integrated for average collision time $\Delta t = (\rho \bar{u})^{-1}$

$$\Delta W = -4\rho m \bar{u}(u_+ - u_-)(\rho \bar{u})^{-1} = -4m(u_+ - u_-)$$

If the fluid in a thermal equilibrium at temperature $T$,

$$\langle u_+ - u_- \rangle = 0, \quad \langle (u_+ - u_-)^2 \rangle = \frac{k_B T}{m}, \quad \langle \Delta W \Delta W \rangle = 16m k_B T = 2D\Delta t \quad \Rightarrow \quad D = 8m k_B T(\rho \bar{u}) = 8\rho m \bar{u} \cdot k_B T = \alpha k_B T$$

fluctuation-dissipation theorem realized!
Chapter 2. STOCHASTIC QUANTIZATION OF SCALAR FIELD THEORY

* The approach of Parisi and Wu: basic observations

Analogy between *Euclidean quantum field theory* and *Classical statistical mechanics*

- Euclidean path integral measure $\Leftrightarrow$ Boltzmann distribution in equilibrium
- Euclidean Green functions $\Leftrightarrow$ correlation functions of statistical systems in equilibrium

$$
\langle \phi(x_1) \cdots \phi(x_n) \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi e^{-\frac{1}{\hbar}S} \phi(x_1) \cdots \phi(x_n), \quad \mathcal{Z} = \int \mathcal{D}\phi e^{-\frac{1}{\hbar}S} \quad (S: \text{Euclidean classical action})
$$

Clearly $\hbar \Leftrightarrow k_B T$ (= 1 in natural unit)

**basic idea**

Realize the Euclidean path integral measure $e^{-\frac{1}{\hbar}S}/\mathcal{Z}$ as an equilibrium distribution of a stochastic process

cf. Monte Carlo simulation: generation of field configurations subject to probability distribution $e^{-\frac{1}{\hbar}S}$ in “equilibrium” on the basis of certain algorithm

- equilibrium = ‘computer time $\to$ infinity limit’
- algorithm = ‘dynamical law w.r.t. computer time’

**strategy of stochastic quantization**

i) Introduction of a fictitious time $^* t$

$$
\phi(x) \longrightarrow \phi(x,t), \quad x = (x_0, x_1, \ldots, x_{n-1}) : \text{Euclidean coordinate in } n \text{ dimensions}
$$

$\phi(x,t)$ is coupled with a fictitious heat reservoir of temperature $k_B T = \hbar = 1$

ii) Dynamics is governed by the Langevin equation

$$
\frac{\partial}{\partial t} \phi(x,t) = -\frac{\delta S[\phi]}{\delta \phi(x)}|_{\phi(x)\rightarrow \phi(x,t)} + \eta(x,t), \quad S[\phi] = \int d^nx L(\phi(x), \partial \phi(x))
$$

$^* x_0$ is the ordinary (but purely imaginary) Euclidean time, not to be confused with the fictitious time $t$. 

---

\[ [\text{Page 12}] \]
or just sloppily expressed as
\[
\frac{\partial}{\partial t} \phi(x, t) = -\frac{\delta S[\phi]}{\delta \phi(x, t)} + \eta(x, t), \quad S[\phi] = \int dtd^n x \mathcal{L}(\phi(x, t), \partial_x \phi(x, t))
\]

Note
\[
\frac{\delta S[\phi]}{\delta \phi(x)} = 0 \iff \text{classical equation of motion}
\]

Gaussian white noise \( \eta(x, t) \)
\[
\langle \eta(x, t) \rangle_\eta = 0, \quad \langle \eta(x, t) \eta(x', t') \rangle_\eta = 2\hbar \delta^n(x - x') \delta(t - t')
\]

and more in general
\[
\langle \eta(x_1, t_1) \cdots \eta(x_k, t_k) \rangle_\eta = \begin{cases} 0 & \text{for odd } k \\ \sum_{\text{all comb.}} \prod_{ij} \langle \eta(x_i, t_i) \eta(x_j, t_j) \rangle_\eta & \text{for even } k \end{cases}
\]

Averages over \( \eta \) are explicitly realized by the distribution
\[
\langle \cdots \rangle_\eta = \mathcal{N}^{-1} \int \mathcal{D} \eta \cdots e^{-\frac{1}{4} \int dtd^n x \eta^2(x, t)}, \quad \mathcal{N} = \int \mathcal{D} \eta e^{-\frac{1}{4} \int dtd^n x \eta^2(x, t)}
\]

iii) Solve the Langevin equation, given an initial condition at \( t = t_0 \), to obtain \( \phi \) as a functional of \( \eta \)
\[
\phi_\eta(x, t) : \text{solution of the Langevin equation, functional of } \eta
\]

The \( k \)-point correlation function reads as
\[
\langle \phi_\eta(x_1, t_1) \cdots \phi_\eta(x_k, t_k) \rangle_\eta = \mathcal{N}^{-1} \int \mathcal{D} \phi_\eta(x_1, t_1) \cdots \phi_\eta(x_k, t_k) e^{-\frac{1}{4} \int dtd^n x \eta^2(x, t)}
\]
iv) Equal (fictitious) time correlation function converges to Euclidean Green function in quantum field theory in $n$-dim. in equilibrium ... main assertion

$$\left. \langle \phi_\eta(x_1, t_1) \cdots \phi_\eta(x_k, t_k) \rangle_{\eta} \right|_{t_1=\cdots=t_k=\infty} \to \frac{1}{Z} \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_k) e^{-S/\hbar}$$

$$Z = \int \mathcal{D}\phi e^{-S/\hbar}$$

In the Fokker–Planck language, introduce the probability distribution functional:* $P(\phi, t)$

equal time correlation function: $$\langle \phi_\eta(x_1, t) \cdots \phi_\eta(x_k, t) \rangle_{\eta} = \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_k) P(\phi, t)$$

Fokker–Planck equation reads as

$$\frac{\partial}{\partial t} P(\phi, t) = \int d^n x \frac{\delta}{\delta \phi(x)} \left( \frac{\delta}{\delta \phi(x)} + \frac{\delta S}{\delta \phi(x)} \right) P(\phi, t)$$

- derivation ... homework(?)

Given an initial distribution, e.g., $P(\phi, 0) = \prod_x \delta(\phi(x))$, the solution is

$$P(\phi, t) = \langle \prod_x \delta(\phi(x) - \phi_\eta(x, t)) \rangle_{\eta} \Leftrightarrow \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_k) P(\phi, t) = \langle \phi_\eta(x_1, t) \cdots \phi_\eta(x_k, t) \rangle_{\eta}$$

Its equilibrium limit reproduces the path integral measure (see later)

$$P(\phi, t) \to e^{-S}/Z \text{ as } t \to \infty$$

* For notational simplicity, no distinction between function and functional will be made. Actually, $P$ is a functional of $\phi$ and function of $t$, so that, e.g., $P[\phi, t]$ would be more appropriate. Such an expression, however, does not look nice and a bit cumbersome to write.
Notes

* Upon introduction of the Fokker–Planck operator

\[ L^* = - \int d^n x \frac{\delta}{\delta \phi(x)} \left( \frac{\delta}{\delta \phi(x)} + \frac{\delta S}{\delta \phi(x)} \right) \]

Fokker–Planck equation \( \dot{P} = -L^* P \) has a formal solution

\[ P(\phi, t) = e^{-L^*(t-t_0)} P(\phi, t_0) \]

Then, \( \forall F \)

\[ \langle F(\phi_\eta(t)) \rangle_\eta = \int D\phi F(\phi) P(\phi, t) = \int D\phi F(\phi) e^{-L^*(t-t_0)} P(\phi, t_0) = \int D\phi P(\phi, t_0) e^{-L(t-t_0)} F(\phi) \]

\[ L = - \int d^n x \left( \frac{\delta}{\delta \phi(x)} - \frac{\delta S}{\delta \phi(x)} \right) \frac{\delta}{\delta \phi(x)} : \text{adjoint operator} \]

* Additional degree of freedom to introduce a kernel \( K \)

\[ \frac{\partial}{\partial t} \phi(x, t) = - \int d^n y K(x, y) \frac{\delta S}{\delta \phi(y, t)} + \eta(x, t) \quad \text{with} \quad \langle \eta(x, t) \rangle_\eta = 0, \quad \langle \eta(x, t) \eta(x', t') \rangle_\eta = 2K(x, x') \delta(t - t') \]

Fokker–Planck equation associated with the kerneled-Langevin equation reads as

\[ \frac{\partial}{\partial t} P(\phi, t) = \int d^n x d^n y \frac{\delta}{\delta \phi(x)} K(x, y) \left( \frac{\delta}{\delta \phi(y)} + \frac{\delta S}{\delta \phi(y)} \right) P(\phi, t) \]

Positive kernel \( K > 0 \) assures the same equilibrium distribution \( \propto e^{-S} \)

- kernel ... control the (fictitious) dynamics toward equilibrium

- Another possibility(?) of introducing two (correlated) white noises \( \eta_1 \) and \( \eta_2 \)

\[ \eta(x, t) = \eta_1(x, t) + \int d^n y K(x, y) \eta_2(y, t), \quad \langle \eta_1(x, t) \eta_2(x', t') \rangle = 2\delta^n(x - x') \delta(t - t') \quad \text{and others } = 0 \]

- kernel can be used for fermion stochastic quantization, stochastic regularization, etc.
2.1 Perturbation theory

The Langevin formulation and stochastic diagrams

Consider a self-interacting (real) scalar field

\[ S = \int d^n x \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{3!} \phi^3 \right) \quad (\mu = 0, \ldots, n - 1) \]

(cubic interaction: just for simplicity and for illustration)

Following the strategy of stochastic quantization, introduce a stochastic field

\[ \phi(x) \rightarrow \phi(x, t) \]

and set up the Langevin equation

\[ \frac{\partial}{\partial t} \phi(x, t) = -\frac{\delta S}{\delta \phi(x, t)} + \eta(x, t) = -(-\partial^2 + m^2) \phi(x, t) - \frac{\lambda}{2!} \phi^2(x, t) + \eta(x, t) \]

\[ \langle \eta(x, t) \rangle_\eta = 0, \quad \langle \eta(x, t) \eta(x', t') \rangle_\eta = 2 \delta^n(x - x') \delta(t - t') \]

In momentum space

\[ \phi(k, t) = \int d^m x e^{ikx} \phi(x, t) \quad \leftrightarrow \quad \phi(x, t) = \int \frac{d^n k}{(2\pi)^n} e^{-ikx} \phi(k, t) \]

\[ \frac{\partial}{\partial t} \phi(k, t) = -(k^2 + m^2) \phi(k, t) - \frac{\lambda}{2!} \int d^np d^nq (2\pi)^n \phi(p, t) \phi(q, t) \delta^n(k - p - q) + \eta(k, t) \]

\[ \langle \eta(k, t) \rangle_\eta = 0, \quad \langle \eta(k, t) \eta(k', t') \rangle_\eta = 2(2\pi)^n \delta^n(k + k') \delta(t - t') \]

Introduce a retarded Green function \( G(k, t) \)

\[ \frac{\partial}{\partial t} G(k, t) = -(k^2 + m^2) G(k, t) + \delta(t), \quad G(k, t) = 0 \quad \text{for} \quad t < 0 \]

\[ \Rightarrow \quad G(k, t) = \theta(t) e^{-(k^2 + m^2)t} \]
Then, solution of the Langevin equation with the initial condition \( \phi(k, 0) = \phi_0(k) \) reads as

\[
\phi(k, t) = e^{-(k^2 + m^2)t} \phi_0(k) + \int_0^\infty dt' G(k, t - t') \left( \eta(k, t') - \frac{\lambda}{2!} \int \frac{d^npd^nq}{(2\pi)^n} \phi(p, t') \phi(q, t') \delta^n(k - p - q) \right)
\]

\[
= e^{-(k^2 + m^2)t} \phi_0(k) + \int_0^t dt' e^{-(k^2 + m^2)(t - t')} \left( \eta(k, t') - \frac{\lambda}{2!} \int \frac{d^npd^nq}{(2\pi)^n} \phi(p, t') \phi(q, t') \delta^n(k - p - q) \right)
\]

... solved iteratively

Actually, since the initial-value contribution decays out exponentially, let \( \phi_0 = 0 \) for simplicity and without losing generality,

\[
\phi(k, t) = \int_0^\infty dt' G(k, t - t') \eta(k, t')
\]

\[
- \frac{\lambda}{2} \int_0^\infty dt' G(k, t - t') \int \frac{d^npd^nq}{(2\pi)^n} \int_0^\infty dt'_1 G(p, t' - t'_1) \eta(p, t'_1) \int_0^\infty dt'_2 G(q, t' - t'_2) \eta(q, t'_2) \delta^n(k - p - q)
\]

\[+ \cdots \]

- symbolically

\[
\phi = \int (G\eta) - \frac{\lambda}{2} \int G \cdot (G\eta)^2 + \left( -\frac{\lambda}{2} \right)^2 \int (G \cdot (G \cdot (G\eta)^2))(G\eta) + G \cdot (G\eta)(G \cdot (G\eta)^2) + \cdots
\]

- g

* line (with an arrow representing time flow \( t \leftarrow t' \) and momentum flow \( k \) : \( G(k, t - t') \)

* cross : noise \( \eta(k, t) \)

* fictitious time is assigned at every end and vertex to be integrated from 0 to \( \infty \) except for the external one \( t \)
Correlation functions at equal fictitious time \( \langle \phi(k_1, t) \cdots \phi(k_\ell, t) \rangle_\eta \)

\( \Leftrightarrow \) make all possible pairs between crosses (= noises):

\[ \langle \eta(k, t) \eta(k', t') \rangle_\eta = 2(2\pi)^n \delta^n(k + k') \delta(t - t') \]

\( \Rightarrow Stochastic \ Diagrams \)
\[ \langle \phi \phi \rangle_\eta = \langle \left( \begin{array}{c} \xrightarrow{} + \xleftarrow{} + \xrightarrow{} + \xrightarrow{} + \xrightarrow{} + o(\lambda^3) \end{array} \right) \rangle_\eta \]

\[ \times \left( \begin{array}{c} \xrightarrow{} + \xrightarrow{} + \xrightarrow{} + \xrightarrow{} + \xrightarrow{} + o(\lambda^3) \end{array} \right) \rangle_\eta \]

\[ = \langle \left( \begin{array}{c} \xrightarrow{}(\xrightarrow{})(\xrightarrow{}(\xrightarrow{})) \end{array} \right) \rangle_\eta + \langle \left( \begin{array}{c} \xleftarrow{}(\xleftarrow{})(\xleftarrow{}(\xleftarrow{})) \end{array} \right) \rangle_\eta + \langle \left( \begin{array}{c} \xleftarrow{}(\xleftarrow{})(\xleftarrow{}(\xleftarrow{})) \end{array} \right) \rangle_\eta \]

\[ + \langle \left( \begin{array}{c} \xleftarrow{}(\xleftarrow{})(\xleftarrow{}(\xleftarrow{})) \end{array} \right) \rangle_\eta \]

\[ + o(\lambda^2) \]

\[ = \xrightarrow{} + 2 \left( \begin{array}{c} \xrightarrow{}(\xrightarrow{})(\xrightarrow{}(\xrightarrow{})) \end{array} \right) \]

\[ + 2 \left( \begin{array}{c} \xleftarrow{}(\xleftarrow{})(\xleftarrow{}(\xleftarrow{})) \end{array} \right) \]

\[ + 2 \left( \begin{array}{c} \xrightarrow{}(\xrightarrow{})(\xrightarrow{}(\xrightarrow{})) \end{array} \right) \]

\[ + 2 \left( \begin{array}{c} \xleftarrow{}(\xleftarrow{})(\xleftarrow{}(\xleftarrow{})) \end{array} \right) + o(\lambda^3) \]
There are so many stochastic diagrams corresponding to a given Feynman diagram in field theory!

Notice there are two different kinds of lines present in the stochastic diagram.

⋆ line with an arrow, i.e., the Green function of the Langevin equation \( G \)

and

⋆ line with a cross, i.e., the lowest-order propagator (2-point correlation function) \( D \)

\[
\langle \phi(k, t) \phi(k', t') \rangle^{(0)}_\eta \equiv D(k; t, t') (2\pi)^n \delta^n(k + k')
\]

\[
= \int_0^t dt_1 \int_0^{t'} dt_2 e^{-(k^2 + m^2)(t - t_1)} e^{-(k'^2 + m^2)(t' - t_2)} \langle \eta(k, t_1) \eta(k', t_2) \rangle_\eta
\]

\[
= 2(2\pi)^n \delta^n(k + k') \int_0^{\min(t,t')} dt_1 e^{-(k^2 + m^2)(t + t')} e^{2(k^2 + m^2)t_1}
\]

\[
= (2\pi)^n \delta^n(k + k') \frac{1}{k^2 + m^2} \left( e^{-(k^2 + m^2)|t-t'|} - e^{-(k^2 + m^2)(t+t')} \right)
\]

Observe: the equal-time correlation function reproduces the Feynman propagator in equilibrium \( t = t' \to \infty \)

\[
D(k; t, t) = \frac{1}{k^2 + m^2} \left( 1 - e^{-2(k^2 + m^2)t} \right) \quad \to \quad \frac{1}{k^2 + m^2} \quad \text{as} \quad t \to \infty
\]

**Remark**

In the stochastic diagram, fictitious time assigned at each vertex has to be integrated, in addition to the ordinary momentum integrations

After such time integrations, collection of all stochastic diagrams (with a given topology) are shown to reproduce the corresponding Feynman diagram in equilibrium!

... Equivalence of stochastic quantization to the ordinary quantization
A perturbative proof of equivalence of the stochastic quantization to the ordinary quantization
—super-transformation invariance of stochastic diagrams and its utilization—


Show explicitly that

- collection of all stochastic diagrams with a given topology converges to

the corresponding Feynman diagram in equilibrium in all orders in perturbation

Key points and observations:

★ Set up the initial condition at $t = -\infty$: $\phi(k, -\infty) = 0$

$$\phi(k, t) = \int_{-\infty}^{\infty} G(k, t - t')dt' \left( \eta(k, t') - \int d^n x e^{ikx} \frac{\delta S_{\text{int}}}{\delta \phi(x, t')} \right) = \int_{-\infty}^{\infty} G(k, t - t')dt' \left( \eta(k, t') - (2\pi)^n \frac{\delta S_{\text{int}}}{\delta \phi(-k, t')} \right)$$

★ Green function $G$ and the propagator (free 2-point correlation function) $D$

$$G(k, t - t') = \theta(t - t') e^{-(k^2 + m^2)(t - t')}$$

$$\langle \phi(k, t)\phi(k', t') \rangle_{\eta}^{(0)} = 2(2\pi)^n \delta^n(k + k') \int_{-\infty}^{\infty} G(k, t - \tau)G(k', t' - \tau) d\tau$$

$$= (2\pi)^n \delta^n(k + k') \frac{1}{k^2 + m^2} e^{-(k^2 + m^2)|t - t'|} \equiv (2\pi)^n \delta^n(k + k') D(k, t - t')$$

... $D$ is now translationally invariant w.r.t. $t \Leftarrow$ already aged (equilibrium) at finite $t$ since initial time is set at $t = -\infty$

$$D(k, t - t') = \frac{1}{k^2 + m^2} e^{-(k^2 + m^2)|t - t'|} \xrightarrow{t = t'} \frac{1}{k^2 + m^2} : \text{Feynman propagator in field theory}$$

★ Lower limit of the internal integration over fictitious time assigned at each vertex is now $-\infty$ (not 0 anymore)

★ Important relation between two functions ($\Rightarrow$ super-transformation invariance)

$$\frac{\partial}{\partial t'} D(k, t - t') = G(k, t - t') - G(k, t' - t)$$
Observe that there is always one and the only one Green function $G$ with an **outgoing time flow** from each vertex ... an important characteristic of stochastic diagrams

Consider a collection of stochastic diagrams corresponding to a Feynman diagram with $E$ external lines, $I$ internal lines and $V$ vertices

There are always $V G$ functions, $(E + I - V) D$ functions (⇐ just the above characteristic of stochastic diagrams)

Amplitude can be written as

$$A = \sum_{\text{all comb.}} \int_{-\infty}^{\infty} dt_1 \cdots dt_V G(t_i - t_1) \cdots G(t_j - t_V) \prod_{\ell \neq m}^{E + I - V} D(t_\ell - t_m)$$

apart from the usual loop momentum integrations (here suppressed) and the numerical factors

- times $t_1, \ldots, t_V$: vertex times to be integrated
- others are external times that are taken to be the same $t$ (⇐ equal time correlation function)

- numerical factors:
  - at each $n$-point vertex with one outgoing $G$, $r$ incoming $G$ and $(n - 1 - r)$ $D$ attached
    $$-g_n \frac{(n - 1)!}{(n - 1)!} \binom{n - 1}{r} r!(n - 1 - r)! = -g_n$$
  - total power of $(2\pi)^n$: $V_k$ $n_k$-point vertices ($V = \sum_k V_k$)
    $$\sum_k V_k (1 - (n_k - 1)) + E + I - V = V - I = 1 - L \iff E + 2I = \sum_k V_k n_k, \quad L - 1 = I - V$$

  ... same as the Feynman diagram! ($L$: number of independent loops)

The amplitude $A$ can be rewritten as

$$A = \int d\xi_1 d\bar{\xi}_1 dt_1 \cdots d\xi_V d\bar{\xi}_V dt_V \prod_{i \neq j}^{E + I} D(t_i - t_j + \bar{\xi}_i \xi_i \theta(t_j - t_i) - \bar{\xi}_j \xi_j \theta(t_i - t_j))$$

Here Grassmann numbers (anti-commuting $c$-numbers) $\xi_i$ and $\bar{\xi}_i$ are introduced at each vertex
- the only nonvanishing integral

\[ \int d\xi_i d\bar{\xi}_i \xi_i = 1 \]

- actually

\[ D(t_i - t_j + \tilde{\xi}_i \xi_i \theta(t_j - t_i) - \tilde{\xi}_j \xi_j \theta(t_i - t_j)) = D(t_i - t_j) + \tilde{\xi}_i \xi_i \tilde{G}(t_j - t_i) + \tilde{\xi}_j \xi_j G(t_i - t_j) \]

⇒ Grassmann numbers at vertices guarantee that only one \( G \) flowing out of each vertex can give nonvanishing contribution after integrations

* Introduce \( G(t, t_1, \tilde{\xi}_1\xi_1) \) through

\[ A = \int d\xi_1 d\bar{\xi}_1 dt_1 G(t, t_1, \tilde{\xi}_1\xi_1) \]

\[ G(t, t_1, \tilde{\xi}_1\xi_1) = \int d\xi_2 d\bar{\xi}_2 dt_2 \cdots d\xi_V d\bar{\xi}_V dt_V \prod_{i \neq j}^{E+I} (D(t_i - t_j) + \tilde{\xi}_i \xi_i \tilde{G}(t_j - t_i) + \tilde{\xi}_j \xi_j G(t_i - t_j)) \]

- \( G \) is invariant under the super transformation

\[ \delta t_1 = (\tilde{\epsilon}_1 + \tilde{\xi}_1\epsilon)\theta(t - t_1), \quad \delta \xi_1 = -\epsilon, \quad \delta \tilde{\xi}_1 = -\tilde{\epsilon} \quad \epsilon, \ \tilde{\epsilon} : \text{infinitesimal Grassmann parameters} \]

**Proof**

\[ \delta G(t, t_1, \tilde{\xi}_1\xi_1) = \int d\xi_2 d\bar{\xi}_2 dt_2 \cdots d\xi_V d\bar{\xi}_V dt_V \sum_{(V_1 - V_\ell)} \delta (D(t_1 - t_\ell) + \tilde{\xi}_1 \xi_1 G(t_\ell - t_1) + \tilde{\xi}_1 \xi_1 \tilde{G}(t_1 - t_\ell)) \]

\[ \times \prod_{i \neq j} (D(t_i - t_j) + \tilde{\xi}_i \xi_i \tilde{G}(t_j - t_i) + \tilde{\xi}_j \xi_j G(t_i - t_j)) \]

\[ \delta (D(t_1 - t_\ell) + \tilde{\xi}_1 \xi_1 G(t_\ell - t_1) + \tilde{\xi}_1 \xi_1 \tilde{G}(t_1 - t_\ell)) \]

\[ = -(\xi_1 \epsilon + \tilde{\xi}_1\epsilon)\theta(t - t_1) \{ G(t_1 - t_\ell) + \tilde{\xi}_1 \xi_1 \tilde{G}(t_1 - t_\ell) \} + \theta(t_1 - t)G(t_\ell - t_1) \]
Now the variation of the integrand of $\mathcal{G}$ reads as

$$-(\bar{\xi}_1 \epsilon + \bar{\epsilon} \xi_1) \sum_{(V_1 - V_{\ell})} \left[ \theta(t - t_1) \{ G(t_1 - t_\ell) + \bar{\xi}_\ell \xi_\ell \frac{\partial}{\partial t_\ell} G(t_1 - t_\ell) \} + \theta(t_1 - t) G(t_\ell - t_1) \right]$$

$$\times \prod_{(V_{\ell} - V_m)} (D(t_\ell - t_m) + \bar{\xi}_\ell \xi_\ell G(t_m - t_\ell) + \bar{\xi}_m \xi_m G(t_\ell - t_m))$$

$$\times \prod_{i \neq j} (D(t_i - t_j) + \bar{\xi}_i \xi_i G(t_j - t_i) + \bar{\xi}_j \xi_j G(t_i - t_j))$$

$$= -(\bar{\xi}_1 \epsilon + \bar{\epsilon} \xi_1) \sum_{(V_1 - V_{\ell})} \sum_{(V_{\ell} - V_m)} \left[ \theta(t - t_1) G(t_1 - t_\ell) \{ G(t_\ell - t_m) + \bar{\xi}_m \xi_m \frac{\partial}{\partial t_m} G(t_\ell - t_m) \} + \theta(t_1 - t) G(t_\ell - t_1) G(t_m - t_\ell) \right]$$

$$\times \prod_{(V_m - V_n)} \left( D(t_m - t_n) + \bar{\xi}_m \xi_m G(t_n - t_m) + \bar{\xi}_n \xi_n G(t_m - t_n) \right) \prod_{i \neq j}' (D(t_i - t_j) + \bar{\xi}_i \xi_i G(t_j - t_i) + \bar{\xi}_j \xi_j G(t_i - t_j))$$

+ terms not dependent on $\bar{\xi}_\ell \xi_\ell$

The procedure can be extended until we reach the external time $t$

- contributions from closed loops vanish, because, e.g., $G(t_1 - t_i) \cdots G(t_j - t_1) = 0$

Finally,

$$\delta \mathcal{G}(t, t_1, \bar{\xi}_1 \xi_1) = -(\bar{\xi}_1 \epsilon + \bar{\epsilon} \xi_1) \int d\xi_2 d\xi_2 dt_2 \cdots d\xi_V d\xi_V dt_V \sum_{(V_1 - V_{\ell})} \sum_{(V_{\ell} - V_m)} \sum_{(V_p - V_q)} \bar{\xi}_\ell \xi_\ell \bar{\xi}_m \xi_m \cdots \bar{\xi}_p \xi_p \bar{\xi}_q \xi_q$$

$$\times \theta(t - t_1) G(t_1 - t_\ell) G(t_\ell - t_m) \cdots G(t_p - t_q) G(t_q - t) \prod_{i \neq j}' (D(t_i - t_j) + \bar{\xi}_i \xi_i G(t_j - t_i) + \bar{\xi}_j \xi_j G(t_i - t_j))$$

$$+ (t_1 \leftrightarrow t)$$

This vanishes because

$$\theta(t - t_1) G(t_1 - t_\ell) G(t_\ell - t_m) \cdots G(t_p - t_q) G(t_q - t) = 0, \quad \theta(t_1 - t) G(t_\ell - t_1) G(t_m - t_\ell) \cdots G(t_q - t_p) G(t - t_q) = 0$$

Thus, $\delta \mathcal{G}(t, t_1, \bar{\xi}_1 \xi_1) = 0$ and $\mathcal{G}(t, t_1, \bar{\xi}_1 \xi_1)$ is invariant under the super transformation!
- \( \mathcal{G} \) is a function of the invariant combination among \( t_1, \bar{\xi}_1, \xi_1 \) under the super transformation

\[
\mathcal{G}(t, t_1, \bar{\xi}_1 \xi_1) = \mathcal{G}(t, t_1 + \theta(t - t_1) \bar{\xi}_1 \xi_1, 0) = \mathcal{G}(t, t_1, 0) + \theta(t - t_1) \bar{\xi}_1 \xi_1 \frac{\partial}{\partial t_1} \mathcal{G}(t, t_1, 0)
\]

- Integrations over \( t_1, \bar{\xi}_1, \xi_1 \) are trivially done to give

\[
A = \int d\xi_1 d\bar{\xi}_1 dt_1 \mathcal{G}(t, t_1, \bar{\xi}_1 \xi_1) = \int_{-\infty}^{\infty} dt_1 \theta(t - t_1) \frac{\partial}{\partial t_1} \mathcal{G}(t, t_1, 0) = \mathcal{G}(t, t, 0)
\]

- The procedure can be easily iterated and we end up with

\[
A = \int d\xi_1 d\bar{\xi}_1 dt_1 \mathcal{G}(t, t_1, \bar{\xi}_1 \xi_1) = \mathcal{G}(t, t, 0)
\]

\[
= \int d\xi_2 d\bar{\xi}_2 dt_2 \cdots d\xi_V d\bar{\xi}_V dt_V \prod_{i \neq j}^{E+1} (D(t_i - t_j) + \bar{\xi}_i \xi_i G(t_j - t_i) + \bar{\xi}_j \xi_j G(t_i - t_j)) \bigg|_{t_1 = t, \bar{\xi}_1 \xi_1 = 0}
\]

\[
= \int d\xi_3 d\bar{\xi}_3 dt_3 \cdots d\xi_V d\bar{\xi}_V dt_V \prod_{i \neq j}^{E+1} (D(t_i - t_j) + \bar{\xi}_i \xi_i G(t_j - t_i) + \bar{\xi}_j \xi_j G(t_i - t_j)) \bigg|_{t_1 = t_2 = t, \bar{\xi}_1 \xi_1 = \bar{\xi}_2 \xi_2 = 0}
\]

\[
\vdots
\]

\[
= \prod_{i \neq j}^{E+1} D(t_i - t_j) \bigg|_{t_i = t_i = t} = \prod_{i}^{E+1} D(k_i, 0)
\]

This is nothing but the corresponding Feynman amplitude!

*Thus, the equivalence of stochastic quantization to the ordinary quantization has been proved in all orders in perturbation.*

**Remark**

- There are other perturbative proofs.
- The proof based on the super transformation invariance is simple and transparent (in my opinion).
- The trick itself may be interesting and might be applicable to other occasions...
2.2 Functional integration approach — Fokker–Planck approach —

* Partition function for Langevin dynamics with a constant kernel $\kappa$

$$Z = \int \mathcal{D}\eta e^{-\frac{1}{4\kappa} \int d^nx dt \eta^2(x,t)} \quad \Leftrightarrow \quad \langle \eta(x,t)\eta(x',t') \rangle_\eta = 2\kappa \delta^n(x-x')\delta(t-t')$$

- Change of variable $\eta \rightarrow \phi$ through the Langevin equation:

$$\partial_t \phi(x,t) = -\kappa \frac{\delta S}{\delta \phi(x,t)} + \eta(x,t)$$

- Limit the fictitious time interval to $[0,t]$ ($t \rightarrow \infty$ limit at the end)

$$Z = \int \mathcal{D}\phi P(\phi(0),0) \det \left[ \frac{\delta \eta}{\delta \phi} \right] \exp \left\{ -\frac{1}{4\kappa} \int_0^t d^nx d\tau \left( \partial_\tau \phi(x,\tau) + \kappa \frac{\delta S}{\delta \phi(x,\tau)} \right)^2 \right\}$$

Functional measure : $\mathcal{D}\phi = \prod_x d\phi(x,\tau)$

- Initial distribution $P(\phi(0),0)$ has been introduced ($\phi(0) \leftarrow \phi(x,0)$ with $x$ being suppressed as a functional), since

$$\langle \phi(x_1,t) \cdots \phi(x_n,t) \rangle = \int \prod_x d\phi(x,0) P(\phi(0),0) \langle \phi_{\eta,\phi(0)}(x_1,t) \cdots \phi_{\eta,\phi(0)}(x_n,t) \rangle_\eta$$

- Jacobian $\det[\delta \eta/\delta \phi]$ contains volume divergences like $\delta^n(0)$ ($\rightarrow$ properly be canceled by counter terms)
  also dependent on the boundary condition ($\rightarrow$ discussion on supersymmetry (later))

- Formally

$$\det \left[ \frac{\delta \eta(x,t)}{\delta \phi(x',t')} \right] = \det \left\{ \partial_t \left( \delta^n(x-x')\delta(t-t') + \kappa \frac{\delta}{\delta \phi(x',t')} \frac{\delta S}{\delta \phi(x,t)} \right) \right\}$$

$$= \exp \left[ \text{Tr} \ln \left\{ \left( \partial_t + \kappa \frac{\partial}{\partial \phi(x,t)} \frac{\delta S}{\delta \phi(x,t)} \right) \delta^n(x-x')\delta(t-t') \right\} \right]$$

$$= \exp \left[ \text{Tr} \ln \left\{ \partial_t \left( \delta(t-t') + \kappa \theta(t-t') \frac{\partial}{\partial \phi(x,t')} \frac{\delta S}{\delta \phi(x,t')} \right) \delta^n(x-x') \right\} \right]$$
here causal propagator for $\partial_t$ is chosen

$$\partial_t(\partial^{-1})_{t,t'} = \delta(t-t') \iff (\partial^{-1})_{t,t'} = a\theta(t-t') - (1-a)\theta(t'-t) \xrightarrow{\alpha=1} \theta(t-t')$$

- apart from irrelevant (field-independent) factor

$$\det[\frac{\delta \eta(x,t)}{\delta \phi(x',t')} ] \propto \exp \left[ \text{Tr} \ln \left\{ \left( \delta(t-t') + \kappa \theta(t-t') \frac{\partial}{\partial \phi(x,t')} \frac{\delta S}{\delta \phi(x,t')} \right) \delta^n(x-x') \right\} \right]$$

$$= \exp \left[ \int_0^t d^n x d t' \kappa \theta(0) \frac{\delta^2 S}{\partial \phi(x)^2} \big|_{\phi(x,t')} \right]$$

$$= \exp \left[ \int_0^t d^n x d t' \frac{\kappa}{2} \frac{\delta^2 S}{\partial \phi(x)^2} \big|_{\phi(x,t')} \right]$$

... confirm this result: home work

- Stratonovich calculus: $\theta(0) = 1/2$ is chosen

 Apparently volume divergence $\delta^n(0)$ appears $\rightarrow$ canceled by other terms in perturbation

 Actually Ito calculus gives trivial Jacobian!

Finally, ($\dot{\phi} \equiv \partial_t \phi$)

$$Z = \int D\phi P(\phi(0),0) \exp \left\{ - \int_0^t d^n x d \tau \left[ \frac{1}{4\kappa} \left( \dot{\phi}(x,\tau) + \kappa \frac{\delta S}{\partial \phi(x,\tau)} \right)^2 - \frac{\kappa}{2} \frac{\delta^2 S}{\partial \phi(x)^2} \big|_{\phi(x,\tau)} \right] \right\}$$

$$= \int D\phi(0) P(\phi(0),0) e^{\frac{1}{2} S(\phi(0))} \int D\phi(t) e^{-\frac{1}{2} S(\phi(t))} \int \tilde{D}\tilde{\phi} e^{-\int_0^t d^n x d \tau L_{F.P.}}$$

where

$$D\phi(0) = \prod_x d\phi(x,0), \quad D\phi(t) = \prod_x d\phi(x,t), \quad \tilde{D}\tilde{\phi} = \prod_{0<\tau<t} d\phi(x,\tau)$$

and

$$L_{F.P.} \equiv \frac{1}{4\kappa} \left( \left( \dot{\phi}(x,\tau) \right)^2 + \kappa^2 \left( \frac{\delta S}{\partial \phi(x,\tau)} \right)^2 \right) - \frac{\kappa}{2} \frac{\delta^2 S}{\partial \phi(x)^2} \big|_{\phi(x,\tau)}$$

: Fokker–Planck Lagrangian(?)

... rederive the above result on the basis of Ito-calculus: home work

27
Connection to Fokker–Planck?

Define

\[ \Psi(\phi, t) \equiv e^{\frac{1}{2}S(\phi)} P(\phi, t), \quad P(\phi, t) : \text{probability distribution} \]

then

\[
\frac{\partial}{\partial t} \Psi(\phi, t) = e^{\frac{1}{2}S(\phi)} \frac{\partial}{\partial t} P(\phi, t)
\]

\[
= e^{\frac{1}{2}S(\phi)} \kappa \int d^n x \left( \frac{\delta}{\delta \phi(x)} \left( \frac{\delta}{\delta \phi(x)} + \frac{\delta S}{\delta \phi(x)} \right) \right) e^{-\frac{1}{2}S(\phi)} \Psi(\phi, t)
\]

\[
= \kappa \int d^n x \left( \frac{\delta}{\delta \phi(x)} - \frac{1}{2} \frac{\delta S}{\delta \phi(x)} \right) \left( \frac{\delta}{\delta \phi(x)} + \frac{1}{2} \frac{\delta S}{\delta \phi(x)} \right) \Psi(\phi, t) \equiv -2\kappa \int d^n x \mathcal{H}_{F.P.} \Psi(\phi, t)
\]

Here Fokker–Planck Hamiltonian density \( \mathcal{H}_{F.P.} \).

\[
\mathcal{H}_{F.P.} = \mathcal{H}_{F.P.}(\phi, \delta/\delta \phi) \equiv \frac{1}{2} \left( -\frac{\delta}{\delta \phi(x)} + \frac{1}{2} \frac{\delta S}{\delta \phi(x)} \right) \left( \frac{\delta}{\delta \phi(x)} + \frac{1}{2} \frac{\delta S}{\delta \phi(x)} \right)
\]

\[
= -\frac{1}{2} \frac{\delta^2}{\delta \phi(x)^2} + \mathcal{U}(\phi)
\]

\[
\mathcal{U}(\phi) = -\frac{1}{4} \frac{\delta^2 S}{\delta \phi(x)^2} + \frac{1}{8} \left( \frac{\delta S}{\delta \phi(x)} \right)^2
\]

· Canonical form corresponding to \( \mathcal{L}_{F.P.} \).

\[
\pi(x, t) = \frac{\partial}{\partial \phi(x, t)} \mathcal{L}_{F.P.} = \frac{1}{2\kappa} \dot{\phi}(x, t)
\]

\[
\mathcal{H} = \pi(x, t) \dot{\phi}(x, t) - \mathcal{L}_{F.P.} = \kappa \left\{ \pi^2(x, t) - \frac{1}{4} \left( \frac{\delta S}{\delta \phi(x, t)} \right)^2 + \frac{1}{2} \frac{\delta^2 S}{\delta \phi(x)^2} \right\}_{\phi(x, t)}
\]

while canonical commutation relation reads (formally \( \hbar \to -i \))

\[
[\phi(x, t), \pi(x', t)] = \delta^n (x - x') \quad \Rightarrow \quad \pi(x, t) = -\frac{\delta}{\delta \phi(x, t)}
\]
and therefore in Schrödinger picture
\[ \frac{\partial}{\partial t} \Psi(\phi, t) = \mathcal{H} \Psi(\phi, t) = \kappa \left\{ \frac{\delta^2}{\delta \phi(x)^2} - \frac{1}{4} \left( \frac{\delta S}{\delta \phi(x)} \right)^2 + \frac{1}{2} \frac{\delta^2 S}{\delta \phi(x)^2} \right\} \Psi(\phi, t) \]

which is exactly the same as above!

\* Generating functional

\[ Z[J] = \int \mathcal{D} \phi(0) P(\phi(0), 0) e^{\frac{1}{2} S(\phi(0))} \int \mathcal{D} \phi(t) e^{-\frac{1}{2} S(\phi(t))} \int \tilde{\mathcal{D}} \dot{\phi} e^{-\int_0^t d^n x d\tau (\mathcal{L}_{F.P.} + J \phi)} \]

\[ \langle \phi(x_1, t_1) \cdots \phi(x_n, t_n) \rangle_\eta = (-1)^n \frac{\delta^n Z[J]}{\delta J(x_1, t_1) \cdots \delta J(x_n, t_n)} \big|_{J=0} \]

⇒ perturbative expansion

- Finally, the equal fictitious time correlation function

\[ \langle \phi(x_1, t) \cdots \phi(x_n, t) \rangle_\eta = \int \mathcal{D} \phi(0) P(\phi(0), 0) e^{\frac{1}{2} S(\phi(0))} \int \mathcal{D} \phi(t) \phi(x_1, t) \cdots \phi(x_n, t) e^{-\frac{1}{2} S(\phi(t))} \int \tilde{\mathcal{D}} \dot{\phi} e^{-\int_0^t d^n x d\tau \mathcal{L}_{F.P.}} \]

has also to be equated to the Fokker–Planck representation

\[ \int \mathcal{D} \phi(t) \phi(x_1, t) \cdots \phi(x_n, t) P(\phi(t), t) \]

\* Example: free scalar field

\[ S = \int d^n x \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) \quad \Rightarrow \quad \mathcal{L}_{F.P.} = \frac{1}{4\kappa} \left( (\dot{\phi}(x, \tau))^2 + \kappa^2 \left( (-\partial^2 + m^2)\phi(x, \tau) \right)^2 \right) - \frac{\kappa}{2} (-\partial^2 + m^2) \delta^n(0) \]

\[ \cdots \text{neglect of field-independent term} \]

in momentum space,

\[ \mathcal{L}_{F.P.} = \frac{1}{4\kappa} |\dot{\phi}(k, \tau)|^2 + \frac{\kappa}{4} (k^2 + m^2)^2 |\phi(k, \tau)|^2 \]
In order to perform functional integration $\mathcal{D}\phi$, write

$$\phi(k, \tau) = \phi_{cl}(k, \tau) + \phi_Q(k, \tau)$$

Assume the initial condition

$$P(\phi(0), 0) = \prod_k \delta(\phi(k, 0))$$

and the classical field

$$\frac{\delta}{\delta \phi(k, \tau)} \left. \int d^nx d\tau \mathcal{L}_{F.P.} \right|_{\phi_{cl}} = 0$$

with the boundary conditions: $\phi_{cl}(k, 0) = 0$, $\phi_{cl}(k, t) = \phi(k, t)$

implying “0-0” boundary condition for the quantum part

$$\phi_Q(k, 0) = \phi_Q(k, t) = 0$$

Classical equation of motion

$$\left[ -\frac{\partial^2}{\partial \tau^2} + \kappa^2 (k^2 + m^2)^2 \right] \phi_{cl}(k, \tau) = 0$$

is solved under the above boundary conditions

$$\phi_{cl}(k, \tau) = \frac{\sinh(k^2 + m^2) \kappa \tau}{\sinh(k^2 + m^2) \kappa t} \phi(k, t)$$

Then the Lagrangian is expanded around the classical field $\partial \mathcal{L}_{F.P.}/\partial \phi_{cl} = 0$

$$\mathcal{L}_{F.P.}(\phi) = \mathcal{L}_{F.P.}(\phi_{cl}) + \frac{1}{2!} \frac{\partial^2 \mathcal{L}_{F.P.}(\phi_{cl})}{\partial \phi_{cl}^2} \phi_Q^2 = \mathcal{L}_{F.P.}(\phi_{cl}) + \frac{1}{4\kappa} \phi_Q^2(k, \tau)(-\partial^2 + \kappa^2 (k^2 + m^2)^2) \phi_Q(k, \tau)$$

... be careful, $\phi(k, \tau)$ is complex $\phi^*(k, \tau) = \phi(-k, \tau)$!

Notice that the functional integration $\mathcal{D}\phi = \mathcal{D}\phi_Q$ gives only a factor independent of $\phi(t)$ ⇒ irrelevant and suppressed
"Classical action" corresponding to \( L_{F.P.} \) is explicitly calculated: since

\[
\int_0^t d\tau L_{F.P.}(\phi_{cl}) = \int_0^t d\tau \left( \frac{1}{4\kappa} \dot{\phi}_{cl}(k, \tau)^2 + \frac{\kappa}{4} (k^2 + m^2)^2 |\phi_{cl}(k, \tau)|^2 \right)
\]

\[
= \frac{1}{4\kappa} \phi_{cl}^*(k, \tau) \phi_{cl}(k, \tau) \bigg|_0^t + \int_0^t d\tau \frac{1}{4\kappa} \phi_{cl}^*(k, \tau) \left( -\partial_\tau^2 + \kappa^2 (k^2 + m^2)^2 \right) \phi_{cl}(k, \tau)
\]

\[
= \frac{1}{4} (k^2 + m^2) |\phi(k, t)|^2 \coth(k^2 + m^2) \kappa t
\]

therefore

\[
\int_0^t d^n x d\tau L_{F.P.}(\phi_{cl}) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{4} (k^2 + m^2) |\phi(k, t)|^2 \coth(k^2 + m^2) \kappa t
\]

We have calculated the RHS of

\[
\langle \phi(x_1, t) \cdots \phi(x_n, t) \rangle_\eta = \int \mathcal{D}\phi(0) P(\phi(0), 0) e^{\frac{1}{2} S(\phi(0))} \int \mathcal{D}\phi(t) \phi(x_1, t) \cdots \phi(x_n, t) e^{-\frac{1}{2} S(\phi(t))} \int \mathcal{D}\phi e^{-\int_0^t d^n x d\tau L_{F.P.}}
\]

for free scalar field and found it to be

\[
\text{RHS} = \int \mathcal{D}\phi(t) \phi(x_1, t) \cdots \phi(x_n, t) e^{-\frac{1}{2} S(\phi(t))} \int \mathcal{D}\phi e^{-\int_0^t d^n x d\tau L_{F.P.}}
\]

\[
= \int \mathcal{D}\phi(t) \phi(x_1, t) \cdots \phi(x_n, t) \exp \left\{ -\frac{1}{2} \int \frac{d^n k}{(2\pi)^n} \frac{1}{2} (k^2 + m^2) |\phi(k, t)|^2 (1 + \coth(k^2 + m^2) \kappa t) \right\}
\]

This is compared with the Fokker–Planck representation and thus the Fokker–Planck distribution reads as

\[
P(\phi, t) = \exp \left\{ -\frac{1}{2} \int \frac{d^n k}{(2\pi)^n} \frac{1}{2} (k^2 + m^2) |\phi(k)|^2 (1 + \coth(k^2 + m^2) \kappa t) \right\} = \exp \left\{ -\int \frac{d^n k}{(2\pi)^n} \frac{1}{2} \phi^*(k) \frac{k^2 + m^2}{1 - e^{-2(k^2 + m^2) \kappa t}} \phi(k) \right\}
\]

\[
\longrightarrow \exp \left\{ -\int \frac{d^n k}{(2\pi)^n} \frac{1}{2} \phi^*(k)(k^2 + m^2) \phi(k) \right\} = e^{-S} \quad \text{as} \quad t \to \infty
\]

- realizes the desired Feynman measure in equilibrium irrespective of the value of constant kernel \( \kappa > 0 \)
- kernel \( \kappa \) controls the speed of relaxation \( \leftrightarrow \) change of fictitious time scale

31
**Fokker–Planck Hamiltonian and its spectrum — nonperturbative proof of equivalence —**

- a simple model of quantum mechanics (finite degrees of freedom) described by a potential $V(x)$ with integrable $e^{-V}$


Fokker–Planck equation

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + \frac{\partial V}{\partial x} \right) P(x,t)$$

Similarity transformation

$$\Psi(x,t) = e^{\frac{1}{2} V(x)} P(x,t), \quad \frac{\partial}{\partial t} \Psi(x,t) = e^{\frac{1}{2} V(x)} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + \frac{\partial V}{\partial x} \right) e^{-\frac{1}{2} V(x)} \Psi(x,t)$$

$$= - \left( - \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial V(x)}{\partial x} \right) \left( \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial V(x)}{\partial x} \right) \Psi(x,t) = -2 H_{F.P.} \Psi(x,t)$$

Fokker–Planck Hamiltonian $H_{F.P.}$ is non-negative, self-adjoint

$$H_{F.P.} = \frac{1}{2} \left( - \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial V(x)}{\partial x} \right) \left( \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial V(x)}{\partial x} \right) = \frac{1}{2} Q^\dagger Q$$

· if $V(x) \geq \alpha x^2 + \text{const.}, \ (\alpha > 0)$, spectrum is known to be purely discrete


Complete set of eigenstates of $H_{F.P.}$

$$H_{F.P.} \Psi_n(x) = E_n \Psi_n(x), \quad E_n \geq 0, \quad n = 0, 1, \ldots$$

can expand $\Psi(x,t)$ as

$$\Psi(x,t) = \sum_{n=0}^{\infty} a_n \Psi_n(x) e^{-2E_n t} = a_0 \Psi_0(x) + \sum_{n=1}^{\infty} a_n \Psi_n(x) e^{-2E_n t}$$

The lowest eigenstate (ground state) $\Psi_0$ belonging to eigenvalue 0 (so it is a stationary state)

$$\Psi_0(x) \propto e^{-\frac{1}{2} V(x)}$$
survives at $t \to \infty$ limit

$$\Psi(x,t) \longrightarrow a_0 e^{-\frac{1}{2}V(x)}$$

and therefore the Fokker–Planck distribution relaxes to

$$P(x,t) \longrightarrow a_0 e^{-V(x)} = \frac{e^{-V(x)}}{\int dx e^{-V(x)}}$$

The desired measure is realized in equilibrium!

- clearly, the existence of nondegenerate discrete 0 is crucial
- generalization to field theory · · · similar, but mathematically needs more rigorous treatment

Chapter 3. GAUGE FIELDS

One of the (original) motivations of stochastic quantization
= to quantize gauge fields without fixing the gauge (or without introducing a gauge fixing term)

3.1 Quick review of quantization of gauge fields and Faddeev–Popov gauge fixing

What’s wrong with quantization of gauge fields?

Of course, nothing is so wrong, but we know ...

∃ gauge invariance ⇔ ∃ redundant degrees of freedom
while
⇔ only physically relevant quantities to be quantized

That is,
limitation to appropriate variables to be quantized is necessary and indispensable
... otherwise disastrous results, like,
- not normalizable Feynman measure $e^{-S}$ ⇔ divergence of gauge volume
- loss of unitarity (non-Abelian gauge field)!

The (ordinary) procedure is

Kill redundant gauge degree of freedom (breaking gauge invariance) by adding a gauge fixing term
→ Canonical or path-integral quantization
→ Nevertheless, physical quantities are gauge independent and the unitarity is preserved

* canonical quantization: Dirac bracket instead of Poisson bracket, physical state condition,...
* path-integral quantization: proper measure, Faddeev-Popov determinant, ghost,...

Let the action $S[A]$ be invariant under the gauge transformation

$$ A_\mu \rightarrow A_\mu^U = UA_\mu U^{-1} - iU \partial_\mu U^{-1}, \quad \det U = 1, \quad UU^\dagger = U^\dagger U = 1 $$

Then $e^{-S[A]}$ is not normalizable!*

* Actually the path-integral measure $\mathcal{D}A$ is gauge invariant!
\[ \int \mathcal{D}A e^{-S[A]} = \infty \quad \text{since} \quad \mathcal{D}A \sim \mathcal{D}U \mathcal{D}A^g \]

A^g : one gauge-fixed \((g[A]|_{A^g} = 0)\) configuration

i.e., infinite overcounting along gauge orbit \(\mathcal{D}U \sim \infty\)

To overcome this, introduce a gauge fixing condition, e.g., \(g[A] = 0\)

\[ \iff \text{most easily be done by inserting } \delta(g[A]) \text{ in the path integral} \]

\textit{But, what is the correct integration measure then??} \quad \Rightarrow \quad \text{Faddeev-Popov procedure}

Define

\[ \Delta^{-1}_g[A] = \int \mathcal{D}U \delta(g[A^U]) \quad \text{or} \quad 1 = \Delta_g[A] \int \mathcal{D}U \delta(g[A^U]) \]

Observe \(\Delta^{-1}_g[A]\) is gauge invariant

\[ \Delta^{-1}_g[A^U] = \int \mathcal{D}U' \delta(g[(A^U)^{U'}]) = \int \mathcal{D}(UU') \delta(g[A^{UU'}]) = \Delta^{-1}_g[A] \]

since \(\mathcal{D}U' = \mathcal{D}(UU')\), i.e., invariant group measure

Then, insert the above identity into the path integral

\[ \int \mathcal{D}A e^{-S[A]} = \int \mathcal{D}A \Delta_g[A] \int \mathcal{D}U \delta(g[A^U]) e^{-S[A]} \]

\[ = \int \mathcal{D}A \Delta_g[A] \int \mathcal{D}U \delta(g[A]) e^{-S[A]} \quad \text{... gauge transf. } A^U \rightarrow A \text{ changes nothing on both sides, except } g[A^U] \rightarrow g[A] \]

\[ = \int \mathcal{D}U \int \mathcal{D}A \Delta_g[A] \delta(g[A]) e^{-S[A]} \]

Thus, the following things have been achieved

- \textit{gauge transf. volume } \int \mathcal{D}U = \infty \text{ is factored out} \]
  \quad \rightarrow \text{infinite overcounting is avoided just by being divided by this factor} \]
- \textit{a gauge-fixed } \(g[A] = 0\) \text{ path integral remains} \]
- \textit{with the correct measure}
So, what is $\Delta_g[A]$?
- from its definition,
\[
\Delta_g^{-1}[A] = \int \mathcal{D}U \delta(g[U]) = \int \mathcal{D}g \det \left( \frac{\partial U}{\partial g} \right) \delta(g) = \det \left( \frac{\partial U}{\partial g} \right) \bigg|_{g=0}, \quad \text{i.e.,} \quad \Delta_g[A] = \det \left( \frac{\partial g[A^U]}{\partial U} \right) \bigg|_{g=0}
\]
in practice, $(g = 0 \Leftrightarrow U = 1)$
\[
U = e^{-i\omega}, \quad \omega \equiv \omega^a(x)\tau^a, \quad \det \left( \frac{\partial g[A^U]}{\partial U} \right) \sim \det \left( \frac{\partial g[A^\omega]}{\partial \omega} \right), \quad A^\omega \sim A + \partial \omega + [A, i\omega]
\]
- determinant (Faddeev–Popov determinant) can be incorporated into the action by means of ghost fields $c$ and $\bar{c}$
\[
\det M = \int \mathcal{D}c \mathcal{D}\bar{c} e^{\int d^n x d^n y c(x)M(x,y)c(y)}
\]
Faddeev–Popov ghost fields $c$ and $\bar{c}$: anti-commuting but scalar fields!
... they never appear in physical-state space, thus no violation of spin-statistics relation

Ex.: in case of $g[A] = \partial \cdot A$
\[
\frac{\partial}{\partial \omega^a(x)} g[A^\omega]_{b}^{v}(y) = \frac{\partial}{\partial \omega^a(x)} \partial_y \cdot (A_{b}^{v}(y) + \partial_y \omega_{b}^{v}(y) + [A, i\omega]_{b}^{v}(y)) = \partial_y \cdot \{(\partial_y \delta_{ab} - f^{abc} A_{c}(y))\delta^n(x - y)\}
\]
\[
\Rightarrow \quad \text{ghost action:} \quad \int d^n x \ c^a(x)(-\partial_x) \cdot (\delta^{ab} \partial_x - f^{abc} A^c(x))c^b(x)
\]
- furthermore, since the original path integral is $g$ independent, an arbitrary weight factor can be introduced, e.g., Gaussian weight
\[
\delta(g) \rightarrow \int \mathcal{D}a \ \delta(g - a) \ e^{-\frac{1}{2\alpha} \int d^n x a^2} = e^{-\frac{1}{2\alpha} \int d^n x g^2[A]} \quad \text{... gauge fixing term, like} \quad \sim \frac{1}{2\alpha} (\partial \cdot A)^2
\]
gauge fixing parameter $\alpha$: 
- Landau gauge \quad $\partial \cdot A = 0$
- Landau gauge \quad $\alpha = 0$
- Feynman gauge \quad $\alpha = 1$

Unfortunately, Gribov problem ... ambiguity present in gauge fixing condition

\[
\text{correspondence } g \ Leftrightarrow U \ \text{not one to one!} \quad \rightarrow \quad \text{singular(!) determinant } \det(\partial U/\partial g), \ \exists \ \text{Gribov copies}
\]

In stochastic quantization of gauge fields, no gauge fixing is required (see later) \Rightarrow free from ghosts and Gribov problem(??)
3.2 Quantization without gauge fixing

Free Maxwell field

Euclidean action $S$

$$ S = \frac{1}{4} \int d^nx F_{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, $$

where $x_0 = ix_0^{(M)}$: Euclidean (purely imaginary) time

$$ A_0 = -iA_0^{(M)} $$

Home work: Clarify the relationship with the Minkowski action

Equation of motion reads

$$ \frac{\delta S}{\delta A_\mu} = 0 = -\partial_\nu F_{\nu\mu} $$

and the Langevin equation

$$ \dot{A}_\mu(x,t) = -\frac{\delta S}{\delta A_\mu(x,t)} + \eta_\mu(x,t) = \partial_\nu F_{\nu\mu}(x,t) + \eta_\mu(x,t) $$

with the Gaussian white noise $\eta_\mu(x,t)$

$$ \langle \eta_\mu(x,t) \rangle_\eta = 0, \quad \langle \eta_\mu(x,t) \eta_\nu(x',t') \rangle_\eta = 2\delta_{\mu\nu}\delta^n(x-x')\delta(t-t') $$

In momentum space,

$$ A_\mu(k,t) = \int d^n x e^{ikx} A_\mu(x,t), \quad \text{etc.} $$

the action reads

$$ S = \int \frac{d^nk}{(2\pi)^n} \frac{1}{2} A_\mu(-k,t) k^2 T_{\mu\nu}(k) A_\nu(k,t), \quad T_{\mu\nu}(k) \equiv \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} $$

and the Langevin equation

$$ \dot{A}_\mu(k,t) = -k^2 T_{\mu\nu}(k) A_\nu(k,t) + \eta_\mu(k,t) $$

37
Notice the projection operators

transversal: \( T_{\mu \nu}(k) = \delta_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \), longitudinal: \( L_{\mu \nu}(k) = \frac{k_\mu k_\nu}{k^2} \), \( T^2 = T, \quad L^2 = L, \quad TL = LT = 0, \quad T + L = 1 \)

Since the retarded Green function, satisfying

\[
(\delta_{\mu \nu} \partial_t + k^2 T_{\mu \nu}(k)) G_{\nu \alpha}(k, t) = \delta_{\mu \alpha} \delta(t)
\]

is solved to

\[
G_{\mu \nu}(k, t) = \theta(t) \left( e^{-k^2 T(k)} \right)_{\mu \nu} = \theta(t) \left( e^{-k^2 t T_{\mu \nu}(k)} + L_{\mu \nu}(k) \right)
\]

the solution to the Langevin equation reads as

\[
A_\mu(k, t) = \int_{-\infty}^{\infty} dt' G_{\mu \nu}(k, t - t') \eta_\nu(k, t') + \left( e^{-k^2 t T_{\mu \nu}(k)} + L_{\mu \nu}(k) \right) c_\nu
\]

Constant vector \( c_\mu \) is fixed by the initial condition \( A_\mu(k, t_0) = A_\mu^{(0)}(k) \) and

\[
A_\mu(k, t) = \int_{t_0}^{t} dt' \left( e^{-k^2 (t-t') T_{\mu \nu}(k)} + L_{\mu \nu}(k) \right) \eta_\nu(t') + \left( e^{-k^2 (t-t_0) T_{\mu \nu}(k)} + L_{\mu \nu}(k) \right) A_\nu^{(0)}(k)
\]

Observe that the initial value survives at \( t \to \infty \)!

Actually, for transverse \( A^{(T)}_\mu(k, t) = T_{\mu \nu}(k) A_\nu(k, t) \) and longitudinal \( A^{(L)}_\mu(k, t) = L_{\mu \nu}(k) A_\nu(k, t) \) components

\[
\dot{A}^{(T)}_\mu(k, t) = -k^2 A^{(T)}_\mu(k, t) + \eta^{(T)}_\mu(k, t)
\]
\[
\dot{A}^{(L)}_\mu(k, t) = \eta^{(L)}_\mu(k, t) \quad \Leftarrow \quad \text{no damping term! random walk}
\]

with nonvanishing two-point noise correlations

\[
\langle \eta^{(T)}_\mu(k, t) \eta^{(T)}_{\nu}(k', t') \rangle_\eta = 2(2\pi)^2 T_{\mu \nu}(k) \delta^n(k + k') \delta(t - t'), \quad \langle \eta^{(L)}_\mu(k, t) \eta^{(L)}_{\nu}(k', t') \rangle_\eta = 2(2\pi)^2 L_{\mu \nu}(k) \delta^n(k + k') \delta(t - t')
\]

No equilibrium is attained at \( t = \infty \) for Maxwell field \( \Rightarrow \) source of trouble?
\[ \langle A_\mu(k,t)A_\nu(k',t')\rangle_\eta = (e^{-k^2(t-t_0)}A_\mu^{(T)}(0)(k) + A_\mu^{(L)}(0)(k))(e^{-k^2(t'-t_0)}A_\nu^{(T)}(0)(k) + A_\nu^{(L)}(0)(k)) \\
+ \int_{t_0}^t dt_1 \int_{t_0}^{t'} dt_2 \langle (e^{-k^2(t-t_1)}\eta_\mu^{(T)}(k,t_1) + \eta_\mu^{(L)}(k,t_1))(e^{-k^2(t'-t_2)}\eta_\nu^{(T)}(k',t_2) + \eta_\nu^{(L)}(k',t_2)) \rangle_\eta \\
= (1 \text{st term}) + 2(2\pi)^n \delta^n(k+k') \int_{t_0}^{\min(t,t')} d\tau (e^{-k^2(t+t'-2\tau)}T_{\mu\nu}(k) + L_{\mu\nu}(k)) \\
= (e^{-k^2(t-t_0)}A_\mu^{(T)}(0)(k) + A_\mu^{(L)}(0)(k))(e^{-k^2(t'-t_0)}A_\nu^{(T)}(0)(k) + A_\nu^{(L)}(0)(k)) \\
+ (2\pi)^n \delta^n(k+k') \left( e^{-k^2|t-t'|} - e^{-k^2(t+t'-2t_0)} \right) T_{\mu\nu}(k) + 2(\min(t,t') - t_0)L_{\mu\nu}(k) \right) \\
\]

The equal time correlation function behaves asymptotically \( t = t' \to \infty \)

\[ \langle A_\mu(k,t)A_\nu(k',t)\rangle_\eta = (e^{-k^2(t-t_0)}A_\mu^{(T)}(0)(k) + A_\mu^{(L)}(0)(k))(e^{-k^2(t-t_0)}A_\nu^{(T)}(0)(k) + A_\nu^{(L)}(0)(k)) \\
+ (2\pi)^n \delta^n(k+k') \left( \frac{1 - e^{-2k^2(t-t_0)}}{k^2} T_{\mu\nu}(k) + 2(t-t_0)L_{\mu\nu}(k) \right) \\
\to A_\mu^{(L)}(0)(k)A_\nu^{(L)}(0)(k') + (2\pi)^n \delta^n(k+k') \left( \frac{1}{k^2} T_{\mu\nu}(k) + 2(t-t_0)L_{\mu\nu}(k) \right) \]

... does not reproduce the ordinary propagator for Maxwell field
- initial longitudinal configuration does not decay out completely!
- linearly divergent \((\propto t \to \infty)\) term in longitudinal component!

... both due to lack of damping term for longitudinal part

Role of initial condition
- only longitudinal component relevant (for transverse part decays out)
Assume
\[ A_{\mu}^{(0)}(k) = L_{\mu\nu}(k)A_{\nu}^{(0)}(k) \Rightarrow set A_{\mu}^{(0)}(k) = \frac{k_{\mu}}{k^2} \phi(k) \text{ (or } k_{\mu}A_{\mu}^{(0)}(k) = \phi(k) \text{)} \]

and Gaussian distribution for \( \phi(k) \)

\[ \mathcal{N}^{-1} e^{-\frac{1}{2\pi} \int \frac{d^nk}{(2\pi)^n} |\phi(k)|^2 \text{ with normalization constant } \mathcal{N} = \int D\phi e^{-\frac{1}{2\pi} \int \frac{d^nk}{(2\pi)^n} |\phi(k)|^2} \]
Remember $\phi^*(k) = -\phi(-k)$ and therefore

$$\overline{\phi(k)\phi(k')^*} \equiv \mathcal{N}^{-1} \int \mathcal{D}\phi \phi(k)\phi(k') e^{-\frac{1}{2\alpha} \int \frac{d^nk}{(2\pi)^n} |\phi(k)|^2} = -\phi(k)\phi^*(-k')^* = -\alpha (2\pi)^n \delta^n(k+k'), \quad \alpha > 0$$

If averaged over initial configurations at $t_0 = 0$ and assume $A^{(T)(0)} = 0$, the equal time 2-point correlation function behaves like

$$\langle A_\mu(k,t)A_\nu(k',t) \rangle_{\eta} = \frac{k_\mu k_\nu'}{k^2 k'^2} \phi(k)\phi(k') + (2\pi)^n \delta^n(k+k') \left( \frac{1-e^{-2k^2 t}}{k^2} T_{\mu\nu}(k) + 2t L_{\mu\nu}(k) \right)$$

$$\rightarrow (2\pi)^n \delta^n(k+k') \left( \frac{1}{k^2} (\delta_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2}) + 2t k_\mu k_\nu \right)$$

- finite part...nothing but the ordinary $\alpha$-gauge propagator!
  ... initial distribution $\rightarrow$ gauge parameter $\alpha$
  - infinite part??

As a matter of fact, we can show that

\[
\text{No divergences in gauge-invariant quantities!}
\]

Observe
- propagator ($A-A$ correlation function) is gauge-dependent
- all gauge-invariant quantities are built up from $F_{\mu\nu}(x)$

Consider gauge-invariant $F-F$ correlation function*

$$\langle F_{\mu\nu}(x,t)F_{\rho\sigma}(x',t) \rangle_{\eta}^\phi = -\int \frac{d^n p d^n q}{(2\pi)^{2n}} \langle (p_\mu A_\nu(p,t) - p_\nu A_\mu(p,t))(q_\rho A_\sigma(q,t) - q_\sigma A_\rho(q,t)) \rangle_{\eta}^\phi e^{-ipx-iqx'}$$

$$= -\int \frac{d^n p d^n q}{(2\pi)^{2n}} \{p_\mu q_\rho \langle A_\nu(p,t)A_\sigma(q,t) \rangle_{\eta}^\phi - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) \} e^{-ipx-iqx'}$$

* $A^{(T)(0)} = 0$ and the distribution $\phi$ for the longitudinal part are assumed as the initial configuration and $\langle \cdots \rangle^\phi \rightarrow \langle \cdots \rangle$ for notational simplicity.
Notice that terms $\propto p_\mu p_\rho L_{\nu\sigma}(p)$ in

$$p_\mu q_\rho \langle A_\nu(p,t)A_\sigma(q,t) \rangle^\phi_\eta = p_\mu q_\rho (2\pi)^2 \delta^n(p+q) \left( \frac{1 - e^{-2p^2t}}{p^2} T_{\nu\sigma}(p) + (2t + \frac{\alpha}{p^2}) L_{\nu\sigma}(p) \right)$$

$$= -(2\pi)^n \delta^n(p+q) p_\mu p_\rho \left( \frac{1 - e^{-2p^2t}}{p^2} \delta_{\nu\sigma} + (2t + \frac{\alpha}{p^2} - \frac{1 - e^{-2p^2t}}{p^2}) L_{\nu\sigma}(p) \right)$$

(they are dependent on the gauge parameter $\alpha$ and divergent $\propto t \to \infty$) cancels out completely in $F$-$F$ correlation function!

$$\langle F_{\mu\nu}(x,t)F_{\rho\sigma}(x',t) \rangle^\phi_\eta = \int \frac{d^np}{(2\pi)^n} \left\{ p_\mu p_\rho \delta_{\nu\sigma} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) \right\} \frac{e^{-ip(x-x')}}{p^2} (1 - e^{-2p^2t})$$

- No divergence, no gauge-parameter dependence remain in gauge-invariant quantity!

* functional formulation for gauge field

Let $F_{GI}$ be a gauge-invariant quantity $F_{GI} = F_{GI}[A^{(T)}]$

$$A_\mu = A_\mu^{(T)} + A_\mu^{(L)}, \quad \eta_\mu = \eta_\mu^{(T)} + \eta_\mu^{(L)}$$

The expectation value of $F_{GI}$ reads as (here $\kappa = 1$)

$$\langle F_{GI} \rangle_t = \int \mathcal{D}\eta^{(T)} \mathcal{D}\eta^{(L)} \ F_{GI}[A^{(T)}(t)] \ e^{-\frac{1}{2}\int d^nxd\tau(\eta^{(T)}(T)^2+\eta^{(L)}(L)^2)}$$

$$= \int \mathcal{D}A(0) P(A(0),0) \ e^{\frac{1}{2}S(A(0))} \ \int \mathcal{D}A(t) \ F_{GI}[A^{(T)}(t)] \ e^{-\frac{1}{2}S(A(t))} \ \int \mathcal{D}\tilde{A} \ e^{-\int_0^t d^nxd\tau \mathcal{L}_{F.P.}}$$

Since

$$S = S(A^{(T)}) = \int \frac{d^nk}{(2\pi)^n} \frac{1}{2} A^{(T)*}(k)k^2 A^{(T)}(k) \quad \text{and} \quad \mathcal{L}_{F.P.} = \frac{1}{4\kappa} (|\dot{A}^{(T)}(k,\tau)|^2 + |\dot{A}^{(L)}(k,\tau)|^2) + \frac{1}{4}(k^2)^2 |A^{(T)}(k,\tau)|^2$$
we have (assume $A^{(T)}(0) = 0$ as before)

$$
\langle F_{GI} \rangle_t = \int \mathcal{D}A^{(L)}(0)P(A^{(L)}(0), 0) \int \mathcal{D}A^{(L)}(t) \int \tilde{\mathcal{D}}A^{(L)} \exp \left\{ -\frac{1}{4\kappa} \int \frac{d^n k}{(2\pi)^n} \int_0^t d\tau |\dot{A}^{(L)}(k, \tau)|^2 \right\} \\
\times \int \mathcal{D}A^{(T)}(t) F_{GI}[A^{(T)}(t)] \exp \left\{ -\frac{1}{2} \int \frac{d^n k}{(2\pi)^n} A^{(T)*}(k, t) \frac{k^2}{1 - e^{-2\kappa k^2 t}} A^{(T)}(k, t) \right\}
$$

... convergent and gauge-parameter independent

**Remarks**

- Limit $t \to \infty$ has to be taken after performing path integrations over $A$

  home work : perform the path integration over $A^{(L)}$ to confirm this

- Why isn’t gauge fixing necessary in stochastic quantization?

  A: introduction of $t$ results in an *invertible* operator

  $$(\partial_\tau \delta_{\mu\nu} + k^2 T_{\mu\nu}(k))^{-1} = G_{\mu\nu}(k, \tau) = \theta(\tau) \left( e^{-k^2 T(k)\tau} \right)_{\mu\nu} = \theta(\tau) \left( e^{-k^2 \tau T_{\mu\nu}(k)} + L_{\mu\nu}(k) \right)$$

  while in the ordinary cases, the operator $k^2 T_{\mu\nu}(k)$ can not be inverted since it is a projection operator

  → necessity to break symmetry!

- The Langevin equation is *gauge-covariant* under the $t$-independent gauge transformation

  $$A_\mu(x, t) \to A_\mu(x, t) + \partial_\mu \chi(x)$$
Chapter 4. SCALAR QED

Scalar QED: a (simplest) nontrivial interacting gauge field theory

* Action

\[
S = \int d^n x \left\{ (D_\mu \phi)^* D_\mu \phi + \frac{1}{4} F_{\mu \nu} F_{\mu \nu} \right\}, \quad D_\mu \phi = (\partial_\mu - ieA_\mu)\phi, \quad (D_\mu \phi)^* = (\partial_\mu + ieA_\mu)\phi^*
\]

* Classical equations of motion

\[
\frac{\delta S}{\delta \phi} = (-\partial_\mu - ieA_\mu)(D_\mu \phi)^* = -D_\mu^* D_\mu^* \phi^* = 0, \quad \frac{\delta S}{\delta \phi^*} = (-\partial_\mu + ieA_\mu)D_\mu \phi = -D_\mu D_\mu \phi = 0
\]

\[
\frac{\delta S}{\delta A_\mu} = -\partial_\nu F_{\nu \mu} + ie(\phi^* D_\mu \phi - (D_\mu \phi)^* \phi) = 0
\]

* The Langevin equations

\[
\dot{\phi} = -\frac{\delta S}{\delta \phi^*} + \eta = D_\mu^2 \phi + \eta = (\partial^2 - ie\partial \cdot A - 2ie A \cdot \partial - e^2 A^2)\phi + \eta
\]

\[
\dot{\phi}^* = -\frac{\delta S}{\delta \phi} + \eta^* = D_\mu^2 \phi^* + \eta^* = (\partial^2 + ie\partial \cdot A + 2ie A \cdot \partial - e^2 A^2)\phi^* + \eta^*
\]

\[
\dot{A}_\mu = -\frac{\delta S}{\delta A_\mu} + \eta_\mu = \partial_\nu F_{\nu \mu} - ie(\phi^* D_\mu \phi - (D_\mu \phi)^* \phi) + \eta_\mu = \partial_\nu F_{\nu \mu} - ie(\phi^* \partial_\mu \phi - 2ie A_\mu \phi^* \phi) + \eta_\mu
\]

with nonvanishing Gaussian white noise correlations (\(\Leftrightarrow\) notice that \(\langle \eta \eta \rangle = 0 = \langle \eta^* \eta^* \rangle\))

\[
\langle \eta(x, t) \eta^*(x', t') \rangle = 2\delta^n(x - x')\delta(t - t'), \quad \langle \eta_\mu(x, t) \eta_\nu(x', t') \rangle = 2\delta_{\mu \nu} \delta^n(x - x')\delta(t - t')
\]

* In momentum space, given the vanishing initial conditions at \(t = 0\), the solutions are

\[
\phi(k, t) = \int_0^t dt' e^{-k^2(t - t')} \left[ \eta(k, t') - e \int \frac{d^np d^np}{(2\pi)^n} \delta^n(k - p - q)(k + q) A_\mu(p, t') \phi(q, t') \right. \\
\left. - e^2 \int \frac{d^np d^np d^nr}{(2\pi)^{2n}} \delta^n(k - p - q - r) A(p, t') A_\mu(q, t') \phi(r, t') \right]
\]
\[ A_\mu(k, t) = \int_0^t dt' (e^{-k_2(t-t')} T_{\mu\nu}(k) + L_{\mu\nu}(k)) \left[ \eta_\nu(k, t') - e \int \frac{d^n p d^n q}{(2\pi)^n} \delta^n(k + p - q) \phi^*(p, t') \phi(q, t')(p + q)_\nu \\
- 2e^2 \int \frac{d^n p d^n q d^n r}{(2\pi)^n} \delta^n(k - n + n - r) A_\nu(p, t') \phi^*(q, t') \phi(r, t') \right] \]

* Vertices:

Noise correlations in momentum space:

\[ \langle \eta(k, t)\eta^*(k', t') \rangle = 2(2\pi)^n \delta^n(k - k') \delta(t - t'), \quad \langle \eta_\mu(k, t)\eta_\nu(k', t') \rangle = 2(2\pi)^n \delta_{\mu\nu} \delta^n(k + k') \delta(t - t') \]
Calculation of \( \langle \phi(k, t)\phi^*(k', t) \rangle \) up to 1-loop order

... to show the roles of transverse and longitudinal components of gauge field

... to confirm that the linear divergence (\( \propto t \to \infty \)) cancels out for a gauge-invariant quantity \( \lim_{x \to y} \langle \phi(x, t)\phi^*(y, t) \rangle \)

Graphically, \( \phi \) and \( A_\mu \) are given by (arrow on the line is omitted for simplicity)
Contributions from the transverse component $A^{(T)}_\mu$

- no divergence when $t \to \infty$

$\Rightarrow$ we are allowed to keep only translationally invariant part in $D$

$$\langle \phi(k, t_i) \phi^*(k', t_j) \rangle^{(0)} = (2\pi)^n \delta^n(k - k') D(k; t_i, t_j), \quad D = \frac{1}{k^2} \left(e^{-k^2|t_i - t_j|} - e^{-k^2(t_i + t_j)}\right) \sim \frac{1}{k^2} e^{-k^2|t_i - t_j|}$$

$$\langle A^{(T)}(k, t_i) A^{(T)}(k', t_j) \rangle^{(0)} = (2\pi)^n \delta^n(k + k') D^{(T)}_{\mu\nu}(k; t_i, t_j), \quad D^{(T)}_{\mu\nu} \sim \frac{T_{\mu\nu}(k)}{k^2} e^{-k^2|t_i - t_j|}$$

$(\leftrightarrow$ contributions from the lower limit of time-integration at vertex can be neglected$)$

- vertex-time integrations are performed according to their time order

external time $= t \geq t_1 \geq \cdots \geq t_k \Rightarrow$ $t_k$-dependence of $G$ and $D$ connected with the vertex $V_k \sim \exp\left\{\sum_{i \in V_k} p_i^2 t_k\right\}$

$\Rightarrow$ integration over $t_k$ gives $\frac{1}{\sum_{i \in V_k} p_i^2} \exp\left\{\sum_{j \in (V_k \cup V_{k-1}) - j \in (V_k \cap V_{k-1})} p_j^2 t_{k-1}\right\}$

$\Rightarrow$ integration over $t_{k-1}$ gives $\frac{1}{\sum_{i \in V_k} p_i^2} \frac{1}{\sum_{j \in (V_k \cup V_{k-1}) - j \in (V_k \cap V_{k-1})} p_j^2} \exp\left\{\sum_{\ell \in V_{k-2} - \cdots} p_\ell^2 t_{k-2}\right\}$

Apart from the usual momentum integrations

$$\int d^np d^nq \delta^n(k - p - q) \delta^n(k' - p - q) = \delta^n(k - k') \int d^np d^nq \delta^n(k - p - q)$$

to be multiplied afterwards, each diagram is easily evaluated to be
(a) \( \propto \frac{(-e)(k + q)_{\mu} T_{\mu\nu}(p)(-e)(q + k')_{\nu}}{p^2 q^2} \left( \frac{1}{k^2 + p^2 + q^2} \frac{1}{k^2 + k'^2} + \frac{1}{k'^2 + p^2 + q^2} \frac{1}{k^2 + k'^2} \right) \)

\( = \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k)_{\nu}}{p^2 q^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \)

(b) \( \propto \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k')_{\nu}}{p^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{k^2 + k'^2} \)

\( = \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k)_{\nu}}{p^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{2k^2} \)

(c) \( \propto \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k')_{\nu}}{q^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{k'^2} \)

\( = \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k)_{\nu}}{q^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{2k^2} \)

(d) \( \propto \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k')_{\nu}}{q^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{k^2 + k'^2} \)

\( = \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k)_{\nu}}{q^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{2k^2} \)

(e) \( \propto \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k')_{\nu}}{q^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{k^2 + k'^2} \)

\( = \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k)_{\nu}}{q^2 k^2} \frac{1}{k^2} \frac{1}{k^2 + p^2 + q^2} \frac{1}{2k^2} \)

Therefore

\( (a)+(b)+(c)+(d)+(e) = \delta^n(k-k') \int d^n p d^n q \delta^n(k-p-q) \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k)_{\nu}}{k^2 + p^2 + q^2} \left( \frac{1}{p^2 q^2} + \frac{1}{p^2 k^2} + \frac{1}{q^2 k^2} \right) \)

\( = \delta^n(k-k') \int d^n p d^n q \delta^n(k-p-q) \frac{e^2 (k + q)_{\mu} T_{\mu\nu}(p)(q + k)_{\nu}}{k^2 p^2 q^2 k^2} \)

... the same result as in the Landau gauge (\( \alpha = 0 \))

Similarly,

\( (f) \propto \frac{-e^2 T_{\mu\mu}(p)}{k^2 + k'^2} \frac{1}{k^2} \frac{1}{k^2 + k'^2} = \frac{-e^2 T_{\mu\mu}(p)}{k^2 p^2} \frac{1}{2k^2}, \quad (g) \propto \frac{-e^2 T_{\mu\mu}(p)}{k^2 + k'^2} \frac{1}{k^2} \frac{1}{2k^2} \)

thus

\( (f)+(g) = \delta^n(k-k') \int d^n p \frac{-e^2 T_{\mu\mu}(p)}{k^2 p^2 k^2} \)

... again, the same result as in the Landau gauge (\( \alpha = 0 \))
Contributions from the longitudinal component $A^{(L)}_{\mu}$

- vertex-time integrations involving longitudinal component $A^{(L)}_{\mu}$ look like

$$\int_{0}^{t_{k-1}} dt_k(t_k)^\ell \exp\left\{ \sum_{i \in V_k} 'p_i^2 t_k \right\} = \frac{1}{\sum_{i \in V_k} p_i^2 (t_{k-1})^\ell} \exp\left\{ \sum_{i \in V_k} 'p_i^2 t_{k-1} \right\} (1 + O(1/t_{k-1}))$$

- power of $t_k \leftarrow \langle A^{(L)}_{\mu}(k, t_{k-1}) A^{(L)}_{\nu}(k', t_k) \rangle = (2\pi)^n \delta^n(k + k') 2t_k L_{\mu\nu}(k)$

- i.e., the leading-order contribution could be read just by replacing each vertex-time $t_k$ in front of $L_{\mu\nu}$ by $t$ (external time)!

$$2t_k L_{\mu\nu}(p) \rightarrow 2t L_{\mu\nu}(p)$$

- momenta in the exponent $\sum'$ ... only scalar's momenta (longitudinal gauge field does not contribute)

Thus, contribution of $A^{(L)}$ to each graph is

(a) $\sim \frac{(-e)(k + q)_{\mu} 2t L_{\mu\nu}(p)(-e)(q + k')_{\nu}}{q^2} \left( \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2} + \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2} \right)$

$$= e^2 \frac{(k + q)_{\mu} 2t L_{\mu\nu}(p)(q + k)_{\nu}}{k^2} \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2}$$

(b) $\sim e^2 \frac{(k + q)_{\mu} 2t L_{\mu\nu}(p)(q + k')_{\nu}}{k^2} \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2} = e^2 \frac{(k + q)_{\mu} 2t L_{\mu\nu}(p)(q + k)_{\nu}}{k^2} \frac{1}{k^2 + q^2} \frac{1}{2k^2}$

(c) $\sim e^2 \frac{(k + q)_{\mu} 2t L_{\mu\nu}(p)(q + k')_{\nu}}{k^2} \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2} = e^2 \frac{(k + q)_{\mu} 2t L_{\mu\nu}(p)(q + k)_{\nu}}{k^2} \frac{1}{k^2 + q^2} \frac{1}{2k^2}$

(d) $\sim e^2 \frac{(k + q)_{\mu} L_{\nu\nu}(p)(q + k')_{\nu}}{q^2 k^2} \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2} = e^2 \frac{(k + q)_{\mu} L_{\nu\nu}(p)(q + k)_{\nu}}{q^2 k^2} \frac{1}{k^2 + q^2} \frac{1}{2k^2}$

(e) $\sim e^2 \frac{(k + q)_{\mu} L_{\nu\nu}(p)(q + k')_{\nu}}{q^2 k^2} \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2} = e^2 \frac{(k + q)_{\mu} L_{\nu\nu}(p)(q + k)_{\nu}}{q^2 k^2} \frac{1}{k^2 + q^2} \frac{1}{k^2 + k'^2}$

i.e., the leading divergent contribution reads

(a)+(b)+(c)+(d)+(e) $\sim \delta^n(k - k') \int d^n p d^n q \delta^n(k - p - q) e^2 \frac{(k + q)_{\mu} 2t L_{\mu\nu}(p)(q + k)_{\nu}}{(k^2 + q^2) k^2} \left( \frac{1}{q^2} + \frac{1}{k^2} \right)$

$$= \delta^n(k - k') \int d^n p d^n q \delta^n(k - p - q) e^2 \frac{(k + q)_{\mu} 2t L_{\mu\nu}(p)(q + k)_{\nu}}{k^2 q^2 k^2}$$

48
Similarly,

\[(f) + (g) \sim \delta^n(k - k') \int d^n p (-e^2) \frac{2tL_{\mu\nu}(p)}{(k^2)^2} \]

\[\star \text{Gauge invariant quantity is free from linear-} t \text{ divergence!} \]

Actually, longitudinal contributions

\[\delta^n(k - k') \int d^n p d^n q \delta^n(k - p - q) e^2 \frac{(k + q)_{\mu} 2tL_{\mu\nu}(p)(q + k)_{\nu}}{k^2 q^2 k^2} + \delta^n(k - k') \int d^n p (-e^2) \frac{2tL_{\mu\nu}(p)}{(k^2)^2} \]

to a \textit{gauge-invariant} quantity \(\lim_{x \to y} \langle \phi(x, t) \phi^*(y, t) \rangle \) reads from

\[\langle \phi(x, t) \phi^*(y, t) \rangle = \int \frac{d^n k d^n k'}{(2\pi)^{2n}} e^{-ikx + ik'y} \langle \phi(k, t) \phi^*(k', t) \rangle \]

\[\sim 2te^2 \int \frac{d^n k d^n k'}{(2\pi)^{2n}} e^{-ikx + ik'y} \delta^n(k - k') \left[ \int d^n p d^n q \delta^n(k - p - q) \frac{(k + q)_{\mu} L_{\mu\nu}(p)(q + k)_{\nu}}{k^2 q^2 k^2} - \int d^n q L_{\mu\nu}(q) \frac{1}{(k^2)^2} \right] \]

\[= 2te^2 \int \frac{d^n k}{(2\pi)^{2n}} \frac{1}{q^2(k^2)^2} \left[ \frac{(k^2 - q^2)^2}{(k - q)^2} - q^2 \right] \]

\[= 2te^2 \int \frac{d^n k d^n q}{(2\pi)^{2n}} e^{-ik(x - y)} \frac{1}{q^2(k^2)^2} \left[ \frac{(k^2 - q^2)^2}{(k - q)^2} - q^2 \right] \]

\[= 2te^2 \int \frac{d^n k d^n q}{(2\pi)^{2n}} e^{-ik(x - y)} \left[ \frac{k^2 - q^2}{q^2 k^2 (k - q)^2} - \frac{2k \cdot (k - q)}{(k^2)^2 (k - q)^2} \right] \]

\[\to 0 \text{ as } x \to y \]

\textbf{Remark}

\[\star \text{(Not only the leading, but) All contributions from longitudinal gauge component vanishes for gauge invariant quantities!} \]

... I have never seen the general proof
5.1 Yang–Mills field: non-Abelian gauge field theory

⋆ Action

\[ S = \int d^nx \mathcal{L}, \quad \mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu, \]

Classical equation of motion

\[ \frac{\delta S}{\delta A^a_\mu(x)} = 0 = -\left( \partial_\nu \delta^{ab} + g f^{abc} A^c_\nu(x) \right) F^b_\nu(x) \equiv -D^a_\nu(x) F^b_\nu(x) \]

⋆ Langevin equation: \( A^a_\mu(x) \rightarrow A^a_\mu(x, t) \)

\[ \dot{A}^a_\mu(x, t) = -\frac{\delta S}{\delta A^a_\mu(x, t)} + \eta^a_\mu(x, t) = D^a_\nu(x, t) F^b_\nu(x, t) + \eta^a_\mu(x, t), \quad \langle \eta^a_\mu(x, t) \eta^b_\nu(x', t') \rangle = 2 \delta^{ab} \delta_\mu_\nu \delta^n(x - x') \delta(t - t') \]

in momentum space

\[ A^a_\mu(k, t) = \int d^n x e^{ikx} A^a_\mu(x, t), \quad A^a_\mu(k, t) = \int \frac{d^n k}{(2\pi)^n} e^{-ikx} A^a_\mu(k, t), \quad \text{etc.} \]

\[ \dot{A}^a_\mu(p, t) = -p^2 T_{\mu\nu}(p) A^a_\nu(p, t) \]

\[ + igf^{abc} \int \frac{d^n q d^n r}{(2\pi)^n} \delta^n(p - q - r)(-q + r)_\nu A^c_\nu(r, t) A^b_\mu(q, t) - q_\nu A^c_\nu(r, t) A^b_\mu(q, t) + q_\mu A^c_\nu(r, t) A^b_\nu(q, t)) \]

\[ + g^2 f^{abc} f^{cde} \int \frac{d^n q d^n r d^n s}{(2\pi)^{2n}} \delta^n(p - q - r - s) A^b_\mu(q, t) A^c_\nu(r, t) A^d_\mu(s, t) + \eta^a_\mu(p, t) \]

\[ \equiv -p^2 T_{\mu\nu}(p) A^a_\nu(p, t) + I^a_\mu(p, t) + \tilde{I}^a_\mu(p, t) + \eta^a_\mu(p, t) \]

- 3-point vertex \( I^a_\mu(p, t) \)

\[ I^a_\mu(p, t) = -\frac{i g f^{abc}}{2} \int \frac{d^n q d^n r}{(2\pi)^n} \delta^n(p + q + r)(\delta_{\mu\alpha}(p - q)_\beta + \delta_{\alpha\beta}(q - r)_\mu + \delta_{\beta\mu}(r - p)_\alpha) A^b_\alpha(-q, t) A^c_\beta(-r, t) \]
- 4-point vertex $\tilde{I}_\mu^a(p,t)$

$$\tilde{I}_\mu^a(p,t) = \frac{g^2 f^{abc} f^{cde}}{2} \int \frac{d^n q d^n r d^n s}{(2\pi)^{2n}} \delta^n(p + q + r + s)(\delta_{\alpha\beta}\delta_{\gamma\mu} - \delta_{\alpha\gamma}\delta_{\beta\mu}) A^b_\alpha(-q,t) A^c_\beta(-r,t) A^d_\gamma(-s,t)$$

$$= \frac{g^2}{6} \int \frac{d^n q d^n r d^n s}{(2\pi)^{2n}} \delta^n(p + q + r + s) \left[ f^{abc} f^{cde} (\delta_{\alpha\beta}\delta_{\gamma\mu} - \delta_{\alpha\gamma}\delta_{\beta\mu}) \right.$$

$$+ f^{ace} f^{dbe} (\delta_{\beta\gamma}\delta_{\alpha\mu} - \delta_{\beta\alpha}\delta_{\gamma\mu})$$

$$+ f^{ade} f^{bce} (\delta_{\gamma\alpha}\delta_{\beta\mu} - \delta_{\gamma\beta}\delta_{\alpha\mu}) \right] A^b_\alpha(-q,t) A^c_\beta(-r,t) A^d_\gamma(-s,t)$$

★ Green function of the Langevin equation ($G_{\mu\nu}^a(k,t) = 0$ for $t < 0$)

$$(\partial_t \delta_{\mu\alpha} + k^2 T_{\mu\alpha}(k)) G_{\alpha\nu}^a(k,t) = \delta^{ab}\delta_{\mu\nu}\delta(t) \rightarrow G_{\mu\nu}^a(k,t) = \theta(t) \delta^{ab} (e^{-k^2 t} T_{\mu\nu}(k) + L_{\mu\nu}(k))$$

... same as in the Abelian case!

the longitudinal component gives

- linear-$t$ divergences which will disappear in gauge invariant quantities (like in Abelian cases)

- finite contributions ⇔ Faddeev–Popov ghost contributions!

◊ These statements have to be verified in gauge-invariant quantities, like

$$\lim_{t \to \infty} \lim_{x \to y} \langle F_{\mu\nu}^a(x,t) F_{\alpha\beta}^b(y,t) \rangle$$

... again, no general proof has been seen so far

cf. M. Namiki et al., Prog. Theor. Phys. 69 (1983) 1580 ... proof up to the second order in perturbation

★ To bypass ghost contributions, consider another Langevin equation

$$\dot{A}_\mu^a(p,t) = -p^2 A_{\mu}^a(p,t) + N_{\mu\nu}(p) \left[ N_{\alpha\nu}(p) \left( I_{\alpha}^a(p,t) + \tilde{I}_{\alpha}^a(p,t) \right) + \eta_{\nu}^a(p,t) \right]$$
where $\forall n_\mu \neq 0$

\[ N_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_\mu n_\nu}{n \cdot p}, \quad N_{\mu\alpha}(p)N_{\nu\alpha}(p) = N_{\mu\nu}(p), \quad N_{\mu\alpha}(p)N_{\nu\alpha}(p) = \delta_{\mu\nu} - \frac{n_\mu p_\nu + p_\mu n_\nu}{n \cdot p} + \frac{p_\mu p_\nu}{(n \cdot p)^2}n^2 \]

The above Langevin equation is pertubatively solved to give

\[ A^a_\mu(p, t) \sim \int_0^t dt' e^{-p^2(t-t')} N_{\mu\nu}(p) [N_{\alpha\nu}(p)(I^a_\alpha(p, t')) + \tilde{I}^a_\alpha(p, t')) + \eta^a_\nu(p, t')] \]

... all components have damping factor $e^{-p^2t} \longrightarrow$ no linear-$t$ divergences(?)

free propagator

\[ \langle A^a_\mu(k, t)A^b_\nu(k', t') \rangle(0) \propto e^{-k^2|t-t'|} - e^{-k^2(t+t')} \frac{1}{k^2} N_{\mu\alpha}(k)N_{\nu\alpha}(k) \sim e^{-k^2|t-t'|} \frac{1}{k^2} \left[ \delta_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{n \cdot k} + \frac{k_\mu k_\nu}{(n \cdot k)^2}n^2 \right] \]

... same as that in axial gauge!

************

Action in an axial gauge is

\[ \int d^n x \left( \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\alpha} (n \cdot A^a)^2 \right) \quad \text{as} \quad \alpha \rightarrow 0 \Rightarrow n \cdot A^a = 0 \]

Free part in momentum space reads as

\[ \frac{1}{2} A^a_\mu(-k)k^2T_{\mu\nu}(k)A^a_\nu(k) + \frac{1}{2\alpha} n_\mu n_\nu A^a_\mu(-k)A^a_\nu(k) \]

Equation of motion

\[ \left( k^2T_{\mu\nu}(k) + \frac{1}{\alpha} n_\mu n_\nu \right) A^a_\nu(k) = 0 \]

Free propagator $\Delta_{\mu\nu}$ satisfies

\[ \Delta_{\mu\sigma}(k) \left( k^2T_{\sigma\nu}(k) + \frac{1}{\alpha} n_\sigma n_\nu \right) = \delta_{\mu\nu} \]
and is given by
\[
\Delta_{\mu\nu}(k) = \frac{1}{k^2} \left[ \delta_{\mu\nu} + \frac{\alpha k^2 + n^2}{(n \cdot k)^2} k_\mu k_\nu - \frac{n_\mu k_\nu + k_\mu n_\nu}{n \cdot k} \right] \xrightarrow{\alpha \to 0} \frac{1}{k^2} \left[ \delta_{\mu\nu} + \frac{n^2}{(n \cdot k)^2} k_\mu k_\nu - \frac{n_\mu k_\nu + k_\mu n_\nu}{n \cdot k} \right]
\]

Faddeev–Popov determinant is field-independent \(\rightarrow\) no ghost contributions!

- in the case of \(g[A] = n \cdot A\)

\[
\frac{\partial}{\partial \omega^a(x)} g[A^\omega]^b(y) = \frac{\partial}{\partial \omega^a(x)} n \cdot (A^b(y) + \partial_y \omega^b(y) + [A, i\omega]^b(y))
\]

\[
= n \cdot (\partial_y \delta^{ab} - f^{abc} A^c(y)) \delta^n(x - y) \xrightarrow{n \cdot A = 0} n \cdot \partial_y \delta^{ab} \delta^n(x - y)
\]

\(\Rightarrow\) ghost action: \(\int d^n x \bar{c}^a(x) (-\delta^{ab} n \cdot \partial_x) c^b(x)\)

**********

\(\star\) The above Langevin equation is expected to give the same perturbative expansion as that in the axial gauge in equilibrium

**Proof**

Observe that the amplitude is written symbolically as (set the initial condition at \(t_0 = -\infty\))

\[
A \sim \sum_{} (NG(t_i - t_1)N)(NG(t_j - t_2)N) \cdots (NG(t_k - t_V)N) \prod_{} \int_{}^{} I + E - V D(t_i - t_j)
\]

where

\[
(NG(t)N) \to \theta(t) e^{-p^2 t} N_{\mu\alpha}(p) N_{\nu\alpha}(p), \quad D(t) \to e^{-p^2 |t|} \frac{1}{p^2} N_{\mu\alpha}(p) N_{\nu\alpha}(p)
\]

Therefore

\[
\partial_t D(t) = -(NG(t)N) + (NG(-t)N)
\]

and a similar argument can be applied to show \(A \sim \prod D(0) \quad \cdots \) No divergences arise!

Note the gauge constraint \(n \cdot A^a = 0\) is accomplished only in equilibrium: \(n \cdot \dot{A}^a = -p^2 n \cdot A^a\)
5.2 Stochastic Gauge Fixing

Gauge transformations depending on the fictitious time

* Alternative formulation for gauge field quantization possible (thanks to the new degree of freedom, i.e., the fictitious time)
  - gauge invariant quantities: unchanged
  - individual stochastic diagrams contributing gauge invariant quantities remain finite for \( t \to \infty \)
    \[ \iff \text{they sum to finite, while each diagram contains divergences, in the previous formulation} \]

basic idea

introduce a (fictitious-time \( t \) dependent) generalized gauge transformation*

\[ A_\mu(x, t) \to B_\mu(x, t) = A_\mu(x, t) - \partial_\mu \Lambda(x, t) \] (Abelian case)

and choose \( \Lambda(x, t) \) so that

- the free propagator remains finite for \( t \to \infty \), while gauge invariant quantities unchanged
- since \( A_\mu^{(T)} = B_\mu^{(T)} \), gauge invariant quantities are irrelevant to the choice of \( \Lambda(x, t) \)
- for longitudinal component,

  a stochastic gauge constraint = a stochastic differential equation that realizes constraint only in equilibrium

  is imposed (\( \iff \exists \) damping force for longitudinal component, but still never gauge fixed and all components are alive!)

Example: Abelian \((U(1))\) case

The Langevin equation in momentum space

\[ \dot{A}_\mu(k, t) = -k^2 T_{\mu\nu}(k) A_\nu(k, t) + \eta_\mu(k, t) \Rightarrow k_\mu \dot{A}_\mu(k, t) = k_\mu \eta_\mu(k, t) \] i.e., longitudinal part ... random walk

If \( t \)-dependent gauge transformed

\[ A_\mu(k, t) \to B_\mu(k, t) = A_\mu(k, t) + ik_\mu \Lambda(k, t) \]

* Remember that the Langevin equation is invariant under the ordinary \( t \)-independent gauge transformation
Langevin equation for $B_\mu$ reads as

$$\dot{B}_\mu(k, t) = -k^2 T_{\mu\nu}(k) (B_\nu(k, t) - i k_\nu \Lambda(k, t)) + \eta_\mu(k, t) + i k_\mu \dot{\Lambda}(k, t) = -k^2 T_{\mu\nu}(k) B_\nu(k, t) + i k_\mu \dot{\Lambda}(k, t) + \eta_\mu(k, t)$$

Now assume a stochastic gauge constraint = a generalization of gauge fixing condition

$$\alpha k_\mu \dot{B}_\mu(k, t) = -k^2 k_\mu B_\mu(k, t) + \alpha k_\mu \eta_\mu(k, t), \quad \alpha > 0$$

- a generalization of the usual Lorentz condition $k_\mu B_\mu = 0$

$$k \cdot B(k, t) = e^{-k^2 (t-t_0)/\alpha} k \cdot B(k, t_0) + \int_{t_0}^{t} dt' e^{-k^2 (t-t')/\alpha} k \cdot \eta(k, t')$$

- implying, as $t \to \infty$,

$$\langle k \cdot B(k, t) \rangle_\eta = e^{-k^2 (t-t_0)/\alpha} k \cdot B(k, t_0) \to 0$$

- but fluctuation remains

$$\langle k \cdot B(k, t) k' \cdot B(k', t) \rangle_\eta \to -\alpha (2\pi)^n \delta^n(k + k')$$

Thus, the gauge function $\Lambda(k, t)$ is determined

$$i \alpha \dot{\Lambda}(k, t) = -k \cdot B(k, t) \quad \Rightarrow \quad \Lambda(k, t) = \frac{i}{\alpha} \int_{t_0}^{t} dt' k \cdot B(k, t') + \Lambda(k, t_0)$$

or, in terms of $A_\mu$,

$$\Lambda(k, t) = e^{-k^2 (t-t_0)/\alpha} \Lambda(k, t_0) + \frac{i}{\alpha} \int_{t_0}^{t} dt' e^{-k^2 (t-t')/\alpha} k \cdot A(k, t')$$

Therefore, the Langevin equation for $B_\mu$

$$\dot{B}_\mu(k, t) = -k^2 T_{\mu\nu}(k) B_\nu(k, t) - \frac{1}{\alpha} k_\mu k_\nu B_\nu(k, t) + \eta_\mu(k, t) = -k^2 \left( T_{\mu\nu} + \frac{1}{\alpha} L_{\mu\nu} \right)(k) B_\nu(k, t) + \eta_\mu(k, t)$$

an invertible operator appears!
its inverse = covariant $\alpha$-gauge propagator!!

$$\left[k^2 \left(T_{\mu\nu} + \frac{1}{\alpha} L_{\mu\nu}\right)(k)\right]^{-1} = \frac{1}{k^2} \left(T_{\mu\nu} + \alpha L_{\mu\nu}\right)(k)$$

- we find a finite propagator in covariant $\alpha$ gauge

$$\langle B_{\mu}(k, t) B_{\nu}(k', t) \rangle_{\eta} \rightarrow (2\pi)^n \delta^n(k + k') \frac{1}{k^2} \left(T_{\mu\nu} + \alpha L_{\mu\nu}\right)(k)$$

Gauge invariance of the Langevin equation for $B_\mu$

- gauge transformation by $\chi(x)$

$$A_\mu^\chi(k, t) = A_\mu(k, t) + ik_\mu \chi(k)$$

transforms $B_\mu$

$$B_{\mu}(k, t) = A_{\mu}(k, t) + ik_\mu \Lambda(k, t) = A_{\mu}(k, t) + ik_\mu \left(e^{-\frac{k^2}{\alpha}(t-t_0)} \Lambda(k, t_0) + \frac{i}{\alpha} \int_{t_0}^{t} dt' e^{-\frac{k^2}{\alpha}(t-t') k \cdot A(k, t')}\right)$$

$$\rightarrow B_{\mu}^\chi(k, t) = A_{\mu}^\chi(k, t) + ik_\mu \left(e^{-\frac{k^2}{\alpha}(t-t_0)} \Lambda(k, t_0) + \frac{i}{\alpha} \int_{t_0}^{t} dt' e^{-\frac{k^2}{\alpha}(t-t') k \cdot A^\chi(k, t')}\right)$$

$$= B_{\mu}(k, t) + ik_\mu \left(\chi(k) - \frac{k^2}{\alpha} \int_{t_0}^{t} dt' e^{-\frac{k^2}{\alpha}(t-t') \chi(k)}\right)$$

$$= B_{\mu}(k, t) + ik_\mu e^{-\frac{k^2}{\alpha}(t-t_0)} \chi(k) \quad \overset{t \to \infty}{\longrightarrow} \quad B_{\mu}(k, t)$$

... $B_\mu$ becomes gauge invariant at $t = \infty$

the Langevin equation remains intact, i.e., form invariant under the (ordinary) gauge transformation

$$\dot{B}_{\mu}^\chi(k, t) = -k^2 \left(T_{\mu\nu} + \frac{1}{\alpha} L_{\mu\nu}\right)(k) B_{\nu}^\chi(k, t) + \eta_\mu(k, t)$$

... confirm the form invariance of the Langevin equation : homework
Observe, for $\alpha \to 0,$

$$k \cdot B(k, t) = 0 \quad \Rightarrow \quad \Lambda(k, t) = \frac{i}{k^2} k \cdot A(k, t) \quad \Rightarrow \quad B_\mu(k, t) = A_\mu(k, t) - \frac{1}{k^2} k \cdot A(k, t) = T_{\mu
u}(k)A_\nu(k, t)$$

i.e., $B = A^{(T)}$ : projection to transverse part

**Functional approach**

- expectation value of gauge invariant quantity $F_{GI} = F_{GI}[A^{(T)}(t)]$ ($t_0 = 0$ and $\kappa = 1$)

$$\langle F_{GI}[A^{(T)}(t)] \rangle = \int \mathcal{D}A^{(T)}(0)P_T[A^{(T)}(0), 0] e^{\frac{i}{2}S[A^{(T)}(0) \int \mathcal{D}A^{(T)}(t) e^{-\frac{i}{2}S[A^{(T)}(t)]}F_{GI}[A^{(T)}(t)]}$$

$$\times \int \tilde{\mathcal{D}}A^{(T)}(\tau) \exp \left\{ - \int_0^\tau d\tau' \frac{d^n k}{(2\pi)^n} L_{FP}(A^{(T)}(k, \tau)) \right\}$$

$$\times \int \mathcal{D}A^{(L)}(0)P_L[A^{(L)}(0), 0] \int \mathcal{D}A^{(L)}(t) \int \tilde{\mathcal{D}}A^{(L)}(\tau) \exp \left\{ - \frac{1}{4} \int_0^\tau d\tau' \frac{d^n k}{(2\pi)^n} | \dot{A}^{(L)}(k, \tau)|^2 \right\}$$

- make a change of variables $A \to B$ (actually, the generalized gauge transformation)

$$A_\mu(k, \tau) = B_\mu(k, \tau) + \frac{k_\mu}{\alpha} \int_0^\tau d\tau' k_\nu B_\nu(k, \tau') \quad \text{i.e.} \quad \begin{cases} A_\mu^{(T)}(k, \tau) = B_\mu^{(T)}(k, \tau) \\ A_\mu^{(L)}(k, \tau) = B_\mu^{(L)}(k, \tau) + \frac{k_\mu}{\alpha} \int_0^\tau d\tau' k_\nu B_\nu^{(L)}(k, \tau') \end{cases}$$

... Jacobians are field-independent $\to$ neglected

- transverse part ... same as before ($P_T[A^{(T)}(0), 0] = \delta(A^{(T)}(0))$)

$$\sim \int \mathcal{D}B^{(T)}(0)P_T[B^{(T)}(0), 0] e^{\frac{i}{2}S[B^{(T)}(0)] \int \mathcal{D}B^{(T)}(t) e^{-\frac{i}{2}S[B^{(T)}(t)]}F_{GI}[B^{(T)}(t)]}$$

$$\times \int \tilde{\mathcal{D}}B^{(T)}(\tau) \exp \left\{ - \int_0^\tau d\tau' \frac{d^n k}{(2\pi)^n} L_{FP}(B^{(T)}(k, \tau)) \right\}$$

$$= \int \mathcal{D}B^{(T)}(t) \exp \left\{ - \frac{1}{2} \int \frac{d^n k}{(2\pi)^n} B^{(T)}(-k, t) \frac{k^2}{1 - e^{-2k^2 t}} B^{(T)}(k, t) \right\} F_{GI}[B^{(T)}(t)]$$
- only longitudinal part needs a further consideration

\[
\int DB^{(L)}(0)PL[B^{(L)}(0),0] \int DB^{(L)}(t) \int \tilde{DB}^{(L)}(\tau) \exp \left\{ -\frac{1}{4} \int_0^t d\tau \frac{d^nk}{(2\pi)^n} |\dot{B}^{(L)}(k,\tau) + \frac{k\mu k\nu}{\alpha} B^{(L)}(k,\tau)|^2 \right\}
\]

\[\Leftarrow \text{observe } k\mu k\nu B^{(L)}(k,t) = k^2 B^{(L)}(k,t)\]

\[\Rightarrow \text{path integral over } B^{(L)} \text{ is similarly done } (PL[B^{(L)}(0),0] = \delta(B^{(L)}(0)))\]

\[= \int DB^{(L)}(t) \exp \left\{ -\frac{1}{2\alpha} \int \frac{d^n k}{(2\pi)^n} B^{(L)}(-k,t) \frac{k^2}{1 - e^{-2k^2 t/\alpha}} B^{(L)}(k,t) \right\}\]

\[\Rightarrow \text{nothing but the } \alpha\text{-gauge fixing term } \frac{1}{2\alpha}(\partial \cdot B)^2!\]

- Actually, as long as gauge-invariant quantities are concerned, we may gauge-transform \(A_\mu \to B_\mu\)

The Langevin equation reads as

\[\dot{B} = -k^2 \left( T + \frac{1}{\alpha} L \right) B + \eta\]

The corresponding Fokker–Planck equation

\[\dot{P}(B,t) = \int d^n x \frac{\delta}{\delta B_\mu(x)} \left( \frac{\delta}{\delta B_\mu(x)} \right) \left( -\left( \delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu + \frac{1}{\alpha} \partial_\mu \partial_\nu \right) B_\nu(x) \right) P(B,t)\]

has the stationary solution

\[P_{st}(B) \propto e^{-S_{st}(B)}, \quad S_{st}(B) = S(B) + \frac{1}{2\alpha} \int d^n x (\partial \cdot B)^2 = \int d^n x \left( \frac{1}{4} F^2(B) + \frac{1}{2\alpha} (\partial \cdot B)^2 \right)\]
Scalar QED with the stochastic gauge fixing

- $t$-dependent gauge transformation $(D_\mu \phi = (\partial_\mu - ieA_\mu)\phi \to e^{ie\Lambda}D_\mu \phi)$ by a specific $\Lambda = \Lambda(B)$

$$A_\mu(x, t) \to A_\mu(x, t) + \partial_\mu \Lambda(x, t) \equiv B_\mu(x, t), \quad \phi(x, t) \to e^{ie\Lambda(x, t)}\phi(x, t) \equiv \phi^B(x, t)$$

- original Langevin equations

$$\dot{\phi} = D^2 \phi + \eta, \quad \dot{\phi}^* = (D^*)^2 \phi^* + \eta^*$$

are written down in terms of $B_\mu, \phi^B, \phi^*$ as

$$\dot{B}_\mu = \partial_\nu F_{\nu \mu}(B) - ie(\phi^B* \to \partial_\mu \phi^B - 2ieB_\mu \phi^B* \phi^B) + \eta + \partial_\mu \dot{\Lambda}$$

$$\dot{\phi}^B = D^2(B)\phi^B + e^{ie\Lambda} \eta + ie\dot{\Lambda}\phi^B, \quad \dot{\phi}^{B*} = (D^*(B))^2 \phi^{B*} + e^{-ie\Lambda} \eta^* - ie\dot{\Lambda}\phi^{B*}$$

- assume a stochastic gauge fixing condition (interacting case)

$$\partial_\mu \dot{B}_\mu = \frac{1}{\alpha} \partial^2 \partial_\mu B_\mu - ie \partial_\mu(\phi^{B*} \to \partial_\mu \phi^B - 2ieB_\mu \phi^B* \phi^B) + \partial_\mu \eta_\mu$$

then we can choose $\Lambda(x, t)$ as

$$\dot{\Lambda} = \frac{1}{\alpha} \partial \cdot B \quad \Rightarrow \quad \Lambda(x, t) = \int_0^t d\tau \frac{1}{\alpha} \partial_\mu B_\mu(x, \tau)$$

so

$$A_\mu(x, t) = B_\mu(x, t) - \frac{1}{\alpha} \partial_\mu \int_0^t d\tau \partial_\nu B_\nu(x, \tau), \quad \phi^B(x, t) = e^{ie \int_0^t d\tau \partial_\nu B_\nu(x, \tau)/\alpha} \phi(x, t)$$

and Langevin equations

$$\dot{B}_\mu = \left(\delta_{\mu\nu} \partial^2 - (1 - \frac{1}{\alpha}) \partial_\mu \partial_\nu\right)B_\nu - ie(\phi^{B*} \to \partial_\mu \phi^B - 2ieB_\mu \phi^B* \phi^B) + \eta_\mu$$
\[ \dot{\phi}^B = D^2(B)\phi^B + \eta^B + \frac{i\epsilon}{\alpha}(\partial \cdot B)\phi^B, \quad \dot{\phi}^{B*} = (D^*(B))^2\phi^{B*} + \eta^{B*} - \frac{i\epsilon}{\alpha}(\partial \cdot B)\phi^{B*} \]

where

\[ \eta^B(x, t) = e^{i\epsilon \int_0^t d\tau \partial_\nu B_\nu(x, \tau)/\alpha} \eta(x, t) \]

and therefore

\[ \langle \eta^B(x, t) \rangle = 0, \quad \langle \eta^B(x, t)\eta^{B*}(x', t') \rangle = \left\langle e^{i\epsilon \int_0^t d\tau \partial_\nu B_\nu(x, \tau)/\alpha} e^{-i\epsilon \int_0^{t'} d\tau' \partial_\nu B_\nu(x', \tau')/\alpha} \right\rangle 2\delta^n(x - x')\delta(t - t') = 2\delta^n(x - x')\delta(t - t') \]

... confirm the equality: home work

- ordinary \((t\text{-independent})\) gauge invariance of the Langevin equations

\[ \dot{\Lambda} = \frac{1}{\alpha} \partial \cdot (A + \partial \Lambda) \quad \Rightarrow \quad \Lambda(x, t) = \frac{1}{\alpha} \int_0^t dt' e^{\alpha^{-1}(t-t')\partial^2 \partial \cdot A(x, t')} \]

and thus

\[ B_\mu(x, t) = A_\mu(x, t) + \partial_\mu \Lambda(x, t) = A_\mu(x, t) + \frac{1}{\alpha} \partial_\mu \int_0^t dt' e^{\alpha^{-1}(t-t')\partial^2 \partial \cdot A(x, t')} \]

gauge transformations by \(\chi(x)\)

\[ A_\mu^\chi(x, t) = A_\mu(x, t) + \partial_\mu \chi(x) \quad \Rightarrow \quad B_\mu^\chi(x, t) = B_\mu(x, t) + \partial_\mu \left(e^{\alpha^{-1}t\partial^2 \chi(x)} \right) \]

\[ (\phi^B)^\chi(x, t) = e^{i\epsilon \int_0^t dt' \partial_\mu B_\mu^\chi(x, t')/\alpha} \phi^\chi(x, t) = \phi^B(x, t) \exp\left\{i\epsilon e^{\alpha^{-1}t\partial^2 \chi(x)} \right\} \]

... leave the Langevin equations intact ← confirm the invariance: home work

- The Fokker–Planck equation

\[ \dot{P}(B, \phi^B, \phi^{B*}, t) = \int d^n x \left[ \frac{\delta}{\delta B_\mu(x)} \left( \frac{\delta}{\delta B_\mu(x)} + \frac{\delta S}{\delta B_\mu(x)} - \frac{1}{\alpha} \partial_\mu \partial_\nu B_\nu(x) \right) \right. \]

\[ + \left. \frac{\delta}{\delta \phi^{B*}(x)} \left( \frac{\delta}{\delta \phi^B(x)} + \frac{\delta S}{\delta \phi^B(x)} + \frac{i\epsilon}{\alpha} \partial_\mu B_\mu(x) \phi^{B*}(x) \right) \right] \]

\[ + \left. \frac{\delta}{\delta \phi^B(x)} \left( \frac{\delta}{\delta \phi^{B*}(x)} + \frac{\delta S}{\delta \phi^{B*}(x)} - \frac{i\epsilon}{\alpha} \partial_\mu B_\mu(x) \phi^B(x) \right) \right] P(B, \phi^B, \phi^{B*}, t) \]

where \(S = S(B, \phi^B, \phi^{B*}) = \int d^n x \left[ |D\phi^B|^2 + \frac{1}{4} F^2(B) \right] \)
Q: Is
\[ P_0 \propto e^{-S - \frac{1}{2}\int d^n x (\partial \cdot B)^2} = e^{-S_{GF}}, \quad S_{GF} : \alpha\text{-gauge fixed action} \]
a stationary solution of the above Fokker–Planck equation?

A: No!
\[ \dot{P}_0 = \frac{ie}{\alpha} \int d^n x (\partial \mu B_\mu(x)) \left( \phi^{B*}(x) \frac{\delta}{\delta \phi^{B*}(x)} - \phi^B(x) \frac{\delta}{\delta \phi^B(x)} \right) P_0 \neq 0 \]
⇒ the choice \( \dot{\Lambda} = \frac{1}{\alpha} \partial \cdot B \) does not imply the ordinary \( \alpha\)\text{-gauge fixed theory!

only for gauge-invariant quantities, \( P_0 \) becomes stationary

\( \diamond \) Which \( \dot{\Lambda} \) would make \( P_0 \propto e^{-S_{GF}} \) a stationary or equilibrium solution?

... open question

Non-Abelian case

Ordinary gauge transformation

\[ A_\mu = A^a_\mu \tau^a, \quad \text{tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab}, \quad [\tau^a, \tau^b] = i f^{abc} \tau^c, \quad U = e^{-igv} = e^{-igv^a \tau^a} \]

\[ A_\mu \rightarrow A^U_\mu = U A_\mu U^{-1} - \frac{i}{g} U \partial_\mu U^{-1} \quad \Rightarrow \quad A^a_\mu \rightarrow A^a_\mu + \partial_\mu v^a + g f^{abc} A^c_\mu v^b + O(g^2) = A^a_\mu + D^{ab}_\mu(A) v^b + O(g^2) \]

* Introduce a \( t \)-dependent gauge transformation \( U(x,t) \)

\[ B_\mu(x,t) \equiv U(x,t) A_\mu(x,t) U^{-1}(x,t) - \frac{i}{g} U(x,t) \partial_\mu U^{-1}(x,t) \]
then
\[ \dot{B}_\mu = \partial_\mu \left( - \frac{i}{g} U \dot{U}^{-1} \right) + ig[B_\mu, - \frac{i}{g} U \dot{U}^{-1}] + U \dot{A}_\mu U^{-1} = D_\mu(B)(- \frac{i}{g} U \dot{U}^{-1}) + U \dot{A}_\mu U^{-1} \]

... confirm the result: homework

- the Langevin equation for \( A_\mu \)

\[ \dot{A}_\mu = -\frac{\delta S}{\delta A_\mu} + \eta_\mu, \quad S = \int d^n x \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} = \int d^n x \frac{1}{2} \text{tr}(F_{\mu\nu} F_{\mu\nu}) \]
- remember 
\[ \frac{\delta S}{\delta A^a_\mu} = -D^{ab}_\nu(A)F^{b\nu}_{\mu}(A) \]

and transforms under gauge transformation \( A \rightarrow A^U \)

\[ D_\nu(A)F^{\nu\mu}(A) \rightarrow D_\nu(A^U)F^{\nu\mu}(A^U) = U(D_\nu(A)F^{\nu\mu}(A))U^{-1} \]

... confirm this: home work

\( B_\mu \) satisfies the Langevin equation

\[ \dot{B}_\mu = U\left(\frac{-\delta S}{\delta A_\mu} + \eta_\mu\right)U^{-1} + D_\mu(B)\left(-\frac{i}{g}U\dot{U}^{-1}\right) = D_\mu(B)F^{\nu\mu}(B) + \eta_\mu U^{-1} + D_\mu(B)\left(-\frac{i}{g}U\dot{U}^{-1}\right) \]

- now

\[ D_\nu(B)F^{\nu\mu}(B) = -\frac{\delta S(B)}{\delta B_\mu} \] \(( \Leftarrow : \text{gauge invariant action } S(A) = S(A^U) = S(B) )\)

- transformed noise \( U\eta_\mu U^{-1} \equiv \eta^B_\mu \): again Gaussian white noise

Note: \( \eta^{ab}_\mu = 2\text{tr}(\tau^a_\mu \eta^b_\mu) = 2\text{tr}(\tau^a U \eta_\mu U^{-1}) = 2\text{tr}(\tau^a U \tau^b U^{-1})\eta^{b}_\mu \)

\[ \langle \eta^B_\mu(x, t) \rangle = \langle 2\text{tr}(\tau^a U(x, t)\tau^b U^{-1}(x, t))\eta^{b}_\mu(x, t) \rangle = 2\langle \text{tr}(\tau^a U(x, t)\tau^b U^{-1}(x, t)) \rangle \langle \eta^{b}_\mu(x, t) \rangle = 0 \]

\[ \langle \eta^{a\nu}_\mu(x, t)\eta^{b\nu}_\mu(x', t') \rangle = 4\langle \text{tr}(\tau^a U(x, t)\tau^c U^{-1}(x, t))\eta^{c}_\mu(x, t)\text{tr}(\tau^b U(x', t')\tau^d U^{-1}(x', t'))\eta^{d}_\nu(x', t') \rangle \]
\[ = 8\delta_{\mu\nu}\delta^{a\nu}(x-x')\delta(t-t')\langle \text{tr}(\tau^a U(x, t)\tau^c U^{-1}(x, t))\text{tr}(\tau^b U(x, t)\tau^c U^{-1}(x, t)) \rangle \]
\[ = 2\delta^{ab}\delta_{\mu\nu}\delta^{a}(x-x')\delta(t-t') \]

... show this result: home work hint: \( (\tau^a)_{ij}(\tau^b)_{ji} = \frac{1}{2}\delta^{ab} \Leftrightarrow (\tau^c)_{ij}(\tau^c)_{k\ell} = \frac{1}{2}\delta_{i\ell}\delta_{jk} \)

- introduce \( \Lambda(x, t) \) by

\[ \Lambda = \frac{i}{g}U\dot{U}^{-1} \]
The Langevin equation for $B_\mu$ now reads as

$$\dot{B}_\mu = -\frac{\delta S}{\delta B_\mu} - \mathcal{D}_\mu(B) \dot{\Lambda} + \eta_\mu^B$$

- A simplest choice of $\dot{\Lambda} = -\alpha^{-1} \partial_\mu B_\mu$ yields

$$\dot{B}_\mu = -\frac{\delta S}{\delta B_\mu} + \alpha^{-1} \mathcal{D}_\mu(B) \partial_\nu B_\nu + \eta_\mu^B$$

... the extra term

$$\alpha^{-1} \mathcal{D}_\mu^a(B) \partial_\nu B_\nu^b = \alpha^{-1} \partial_\mu \partial_\nu B_\nu^a + \frac{\alpha}{\alpha} f^{abc} \partial_\nu B_\nu^b B_\mu^c$$

contributes to yield covariant $\alpha$-gauge propagator and induces an extra 3-point vertex $J_\mu^a$

The Langevin equation in momentum space

$$\dot{B}_\mu^a(p, t) = -p^2 \left( \delta_{\mu\nu} - (1 - \alpha^{-1}) \frac{p_\mu p_\nu}{p^2} \right) B_\nu^a(p, t) + I_\mu^a(p, t) + \tilde{I}_\mu^a(p, t) + J_\mu^a(p, t) + \eta_\mu^a B(p, t)$$

- extra 3-point vertex (not present in the ordinary treatment!)

$$J_\mu^a(p, t) = \frac{ig}{\alpha} \int \frac{d^n q d^n r}{(2\pi)^n} \delta^n(p + q + r) f^{abc} q_\nu B_\nu^b(-q, t) B^c_\mu(-r, t)$$

$$= \frac{ig}{2\alpha} \int \frac{d^n q d^n r}{(2\pi)^n} \delta^n(p + q + r) f^{abc} (q_\alpha \delta_{\beta\mu} - r_\beta \delta_{\alpha\mu}) B^b_\alpha(-q, t) B^c_\beta(-r, t)$$

... only partially (anti-)symmetric among $q$ and $r$

... home work: check if this extra term $\sim f^{abc} \partial_\nu B_\nu^b B_\mu^c$ satisfies integrability condition or not

* Clearly, vertices $I$ and $\tilde{I}$ will yield corresponding Feynman diagrams in covariant $\alpha$ gauge
- no divergences expected
- but would lose unitarity if no other contributions present $\Leftrightarrow$ absence of ghost contribution
- The extra vertex $J$ should (correctly, at least, for gauge-invariant quantities) produce ordinary Faddeev–Popov ghost contributions!

**Exercise**: calculate the contributions from $J$-vertex to $\langle B^a_\mu B^b_\nu \rangle$ up to one-loop order and compare them with the ordinary Faddeev–Popov ghost contribution and show that they coincide to each other in a gauge-invariant quantity like $\langle FF \rangle$.

**Remarks**
- No general proof has been proposed so far (at least, I don’t know such a proof)
- It seems *not* compatible to use an arbitrary kernel with the stochastic gauge fixing ... only gauge-covariant kernel allowed?

**Functional approach**

⋆ Generating functional

$$Z[J] = \int D\eta B e^{-\frac{1}{4} \int d^n x dt (\eta^a_\mu B^a_\mu(x,t))^2 + \int d^n x dt J^a_\mu(x,t) B^a_\mu(x,t)}$$

$$\propto \int DB \exp \left\{ \int d^n x dt \left( -\frac{1}{4} \left( \dot{B}^a_\mu(x,t) + \frac{\delta S}{\delta B^a_\mu(x,t)} \right) - \alpha^{-1} D^a_\mu(B) \partial \cdot B^b(x,t) \right)^2 + \frac{1}{2} \frac{\delta}{\delta B^a_\mu(x,t)} \left( \frac{\delta S}{\delta B^a_\mu(x,t)} - \alpha^{-1} D^a_\mu(B) \partial \cdot B^b(x,t) \right) \right\}$$

... show this result: home work

- perturbative expansion based on $\mathcal{L}_{FP}$
  ... possible, but rather involved ($\leftarrow$ stochastic gauge fixing term & volume divergence $\propto \delta^n(0)$)

⋆ Fokker–Planck equation

$$\dot{P}(B,t) = \int d^n x \frac{\delta}{\delta B^a_\mu(x)} \left( \frac{\delta}{\delta B^a_\mu(x)} + \frac{\delta S}{\delta B^a_\mu(x)} - \alpha^{-1} D^a_\mu(B) \partial \cdot B^b(x,t) \right) P(B,t)$$

- no exponential (stationary or equilibrium) solution has been found so far
  ... here perturbative treatment is presented

$$S = S_0 + S_1 + S_2, \quad P = \sum_{k=0}^\infty g^k P_k$$
recursion relations among \( P_k \)s follow

\[
\dot{P}_k = \int d^n x \frac{\delta}{\delta B^a_\mu(x)} \left( \frac{\delta}{\delta B^a_\mu(x)} + \frac{\delta S_0}{\delta B^a_\mu(x)} - \alpha^{-1} \partial_\mu \partial_\nu B^a_\nu(x) \right) P_k \\
+ \int d^n x \frac{\delta}{\delta B^a_\mu(x)} \left( \frac{\delta S_1}{\delta B^a_\mu(x)} - \frac{g}{\alpha} \frac{\sqrt{a}}{a_{\mu}} \partial_\nu B^b_\mu(x) B^c_\nu(x) \right) P_{k-1} + \int d^n x \frac{\delta}{\delta B^a_\mu(x)} \frac{\delta S_2}{\delta B^a_\mu(x)} P_{k-2}
\]

\( k = 0 \) (the lowest) case

\[
\dot{P}_0 = \int d^n x \frac{\delta}{\delta B^a_\mu(x)} \left( \frac{\delta}{\delta B^a_\mu(x)} + \frac{\delta S_0}{\delta B^a_\mu(x)} - \alpha^{-1} \partial_\mu \partial_\nu B^a_\nu(x) \right) P_0 \\
= \int \frac{d^n k}{(2\pi)^n} \frac{\delta}{\delta B^a_\mu(k)} \left( \frac{\delta}{\delta B^a_\mu(-k)} + k^2 (T_{\mu\nu}(k) + \alpha^{-1} L_{\mu\nu}(k)) B^a_\nu(k) \right) P_0 \equiv H[B] P_0
\]

introduce \( K \) by

\[
P_0[B, t] = \int DB' K[B, t; B', 0] P_0[B', 0] \Rightarrow \dot{K}[B, t; B', 0] = H[B] K[B, t; B', 0] \text{ with } K[B, 0; B', 0] = \delta[B - B']
\]

then formal solution (suppress group indices and integration, for notational simplicity)

\[
K[B, t; B', 0] = e^{H[B]t} \delta[B - B'] = e^{H[B']t} \delta[B - B'] e^{-H[B']t} = \delta[B - B'(t)]
\]

\[
H[B'] = \int \left( \frac{\delta}{\delta B^a_\mu'} - k^2 (T_{\mu\nu} + \alpha^{-1} L_{\mu\nu}) B^a_\nu' \right) \frac{\delta}{\delta B^a_\mu'}
\]

\( B' \) is transformed to \( B'(t) = e^{H[B']t} B' e^{-H[B']t} \)

\[
B'(t) = e^{-k^2(T + \alpha^{-1} L)t} B' + \frac{2}{k^2} (T + \alpha^{-1} L)^{-1} \sinh\left\{ k^2(T + \alpha^{-1} L)t \right\} \frac{\delta}{\delta B'}
\]

... derive this result : home work
\[ \delta[B - B'(t)] = \int \mathcal{D} \left( \frac{\lambda}{2\pi} \right) e^{i\lambda(B - B'(t))} \]

\[ = \int \mathcal{D} \left( \frac{\lambda}{2\pi} \right) \exp \left\{ -\lambda \frac{1}{2k^2} (T + \alpha^{-1}L)^{-1}(1 - e^{-2k^2(T + \alpha^{-1}L)t}) \lambda + i\lambda(B - e^{-k^2(T + \alpha^{-1}L)t}B') \right\} \]

\[ \propto \exp \left\{ -(B - e^{-k^2(T + \alpha^{-1}L)t}B') \frac{k^2}{2} (T + \alpha^{-1}L)(1 - e^{-2k^2(T + \alpha^{-1}L)t})^{-1}(B - e^{-k^2(T + \alpha^{-1}L)t}B') \right\} \]

the exponent is

\[ -\frac{1}{2} \tilde{B} \Delta^{-1} \tilde{B} \equiv -\frac{\tilde{B} k^2}{2} (T + \alpha^{-1}L)(1 - e^{-2k^2(T + \alpha^{-1}L)t})^{-1} \tilde{B} = -\frac{1}{2} \tilde{B} k^2 ((1 - e^{-2k^2t})^{-1} T + \alpha^{-1}(1 - e^{-2\alpha^{-1}k^2t})^{-1} L) \tilde{B} \]

thus

\[ \Delta = \frac{1}{k^2} ((1 - e^{-2k^2t})T + \alpha(1 - e^{-2\alpha^{-1}k^2t})L) \quad \text{or} \quad \Delta_{\mu\nu}^{ab}(k, t) = \frac{\delta_{ab}}{k^2} ((1 - e^{-2k^2t})T_{\mu\nu}(k) + \alpha(1 - e^{-2\alpha^{-1}k^2t})L_{\mu\nu}(k)) \]

finally (\( P_0[B', 0] \sim \delta[B'] \) for simplicity)

\[ P_0[B, t] \propto \exp \left\{ -\frac{1}{2} \int \frac{d^n k}{(2\pi)^n} B_{\mu}^a(-k) \Delta_{\mu\nu}^{ab}(k, t) B_{\nu}^b(k) \right\} \]

**Properties of the stochastic gauge fixing force**

* stochastic gauge fixing (or Zwanziger) term : \( \mathcal{D}_{\mu}^{ab}(B) \dot{\Lambda}^b \) (e.g., \( \dot{\Lambda}^b = -\partial \cdot B^b \))

- in general, it is a **non-conservative** force

\[ \frac{\delta}{\delta B_{\mu}^a(x, t)} D_{\nu}^{bc}(B) \dot{\Lambda}^c(x', t') - \frac{\delta}{\delta B_{\nu}^b(x', t')} D_{\mu}^{ac}(B) \dot{\Lambda}^c(x, t) \]

\[ = 2\delta_{\mu\nu}gf^{abc} \dot{\Lambda}^c(x, t) \delta^n(x - x') \delta(t - t') + gf^{bce} B_{\nu}^c(x', t') \frac{\delta \dot{\Lambda}^c(x', t')}{\delta B_{\mu}^a(x, t)} - gf^{ace} B_{\mu}^e(x, t) \frac{\delta \dot{\Lambda}^c(x, t)}{\delta B_{\nu}^b(x', t')} \neq 0 \]

66
... it can not be written as a gradient of some action

\[ \exists S_{SGF} \text{ s.t., } \mathcal{D}_{\mu}^{ab}(B)\dot{\Lambda}^b = \frac{\delta S_{SGF}}{\delta B_{\mu}} \]

- When \( \dot{\Lambda} = -\alpha^{-1}\partial \cdot B \) is chosen \( \Rightarrow \) stochastic gauge fixing force = \( \alpha^{-1}\mathcal{D}_{\mu}^{ab}(B)\partial \cdot B^b \)
  i) for small \( \alpha \sim 0 \) (i.e., deterministic case)

  \[
  \dot{B}_{\mu}^a = \alpha^{-1}\mathcal{D}_{\mu}^{ab}(B)\partial \cdot B^b
  \]

  it is a restoring force toward the origin \( B = 0 \)

... in fact, introduce a norm (note that the distance \( ||B - B'|| \) is gauge invariant)

\[
||B||^2 \equiv \int d^nx B_{\mu}^a(x,t)B_{\mu}^a(x,t) = (B,B) \geq 0
\]

then

\[
\frac{d}{dt}||B||^2 = 2(B, \dot{B}) = 2\alpha^{-1}(B, \mathcal{D}(B)\partial \cdot B) = -2\alpha(\partial \cdot B, \partial \cdot B) = -2\alpha||\partial \cdot B||^2 \leq 0
\]

Define Faddeev–Popov operator

\[
L(B) \equiv -\partial_{\mu} \mathcal{D}_{\mu}(B) \quad \Leftrightarrow \quad S_{\text{ghost}} = \int d^n x \bar{c}^a(-\partial_{\mu} \mathcal{D}_{\mu}^{ab}(B))c^b
\]

and introduce Gribov region \( \Omega \) (an open, convex, bounded set, chosen to include the origin \( B = 0 \))

... a hyperplane where \( \partial \cdot B = 0 \) and the operator \( L(B) \) non-negative (i.e., all its eigenvalues non-negative)

\( \rightarrow \) Gribov horizon \( \partial\Omega \) ... characterized by zero eigenvalue of \( L(B) \)

Consider a vicinity of \( \partial \cdot B = 0 : B = B^{(0)} + \epsilon B^{(\perp)} \) with \( \partial \cdot B^{(0)} = 0 \)

since

\[
\frac{d}{dt}||\partial \cdot B||^2 = 2(\partial \cdot B, \partial \cdot \dot{B}) = -\frac{2}{\alpha}(\partial \cdot B, L(B)\partial \cdot B) \sim -\frac{2}{\alpha}\epsilon^2(\partial \cdot B^{(\perp)}, L(B^{(0)})\partial \cdot B^{(\perp)})
\]
- $B^{(0)} \in \Omega$ where $L(B^{(0)}) \geq 0$ is a stable fixed point against stochastic gauge fixing force $\iff \frac{d}{dt} ||\partial \cdot B||^2 \leq 0$
- $B^{(0)} \not\in \Omega$

ii) with drift force and noise term (i.e., for not necessarily small $\alpha$)

in perturbative region (near the origin $B = 0$) : following scenario is expected

a ... in the vicinity of $\Omega$, restoring force exists : $B \to \Omega$

b ... but very close to $\Omega$, restoring force becomes weak while noise term occasionally kicks $B$ out of $\Omega$

c ... outside $\Omega$, stochastic gauge fixing force acts to make $B$ repelled from $\partial \cdot B = 0$ hyperplane

d ... but finally, $B$ would be attracted to the origin (?)
diamond a toy model \((B \Rightarrow x, \partial \cdot B \Rightarrow y)\)

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = -\frac{1}{\alpha} \begin{pmatrix}
2xy^2 \\
(1-x^2)y
\end{pmatrix} \Leftrightarrow \frac{\partial}{\partial y} F_x = -\frac{4}{\alpha} xy \neq \frac{2}{\alpha} xy = \frac{\partial}{\partial x} F_y : \text{non-conservative force!}
\]

- it’s a restoring force, though not conservative

\[
\frac{d}{dt} (x^2 + y^2) = -\frac{1}{\alpha} (1 + x^2)y^2 \leq 0
\]

- ‘Gribov region’ \(\Omega\) can be defined, since

\[
\frac{d}{dt} y^2 = -\frac{2}{\alpha} (1-x^2)y^2 \rightarrow \Omega = \{(x, y); y = 0, |x| \leq 1\}
\]

- ‘Gribov horizon’ = \((-1, 0), (1, 0)\)

all solutions flow into the origin \((0, 0)\)

\[
\frac{dy}{dx} = \frac{1 - x^2}{2xy} \Rightarrow \text{solution} : 2y^2 = \ln x^2 - x^2 + \text{const.}
\]

**Remark**

- stochastic gauge fixing term \(\dot{\Lambda}\), realizing equilibrium Faddeev–Popov distribution, is known to be *non-local!*
- such non-local property might resolve Gribov problem?? ... open question (?) at least, I don’t know its resolution)
Chapter 6. FERMIONS

★ Stochastic quantization of higher spin (e.g., \( s = 2 \)) fields
   
   \( \exists \) higher symmetry (gauge invariance, reparametrization invariance, ...)
   
   \( \Rightarrow \) more involved (e.g, noises with appropriate Lorentz indices)
   
   still the main idea would be valid:

   stochastic process in \( n + 1 \)-dim. reproduces \( n \)-dim. field theory in equilibrium

Ex.: tensor and (bosonic) string field theories have already been examined ... but will not be treated here

★ Fermions?
   
   - no classical analogue \( \Rightarrow \) how to proceed??
   
   ... introduction of anti-commuting \( c \)-numbers (Grassmann numbers)
   
   \( Q \) : Which Langevin equation has to be set up? Equilibrium??

6.1 Naive Fermionic Langevin Equation

★ Let \( S[\psi, \bar{\psi}] \) be an action in Euclidean coordinates \( x \)
   
   - Introduce Grassmann stochastic variables \( \psi(x,t) \) and \( \bar{\psi}(x,t) \) and consider the Langevin equations

\[
\frac{\partial}{\partial t} \psi(x,t) = -\frac{\delta S}{\delta \bar{\psi}(x,t)} + \eta(x,t), \quad \frac{\partial}{\partial t} \bar{\psi}(x,t) = \frac{\delta S}{\delta \psi(x,t)} + \bar{\eta}(x,t)
\]

... here left-derivative convention adopted

with anti-commuting noises

\[
\langle \eta_\alpha(x,t) \rangle = \langle \bar{\eta}_\alpha(x,t) \rangle = 0, \quad \langle \eta_\alpha(x,t) \eta_\beta(x',t') \rangle = \langle \bar{\eta}_\alpha(x,t) \bar{\eta}_\beta(x',t') \rangle = 0 \]
\[
\langle \eta_\alpha(x,t) \bar{\eta}_\beta(x',t') \rangle = 2\delta_{\alpha\beta}\delta^n(x-x')\delta(t-t')
\]

... once \( S \) is given, perturbative treatment should be the same as in the bosonic cases
   
   but it is not clear if the proper equilibrium is attained at \( t = \infty \)
- **Free field case**
  - **Euclidean action**
    \[
    S = \int d^nx \bar{\psi}(-i\gamma_\mu \partial_\mu + m)\psi \quad \Leftrightarrow \quad iS^{(M)} = i \int d^nx \bar{\psi}(i\gamma_\mu^{(M)} \partial_\mu - m)\psi
    \]
    \[
    \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}
    \]
    \[
    \{\gamma_\mu^{(M)}, \gamma_\nu^{(M)}\} = 2g_{\mu\nu}, \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1)
    \]
    ... confirm the correspondence between Euclidean and Minkowski formulations: home work

- **Langevin equations**
  \[
  \frac{\partial}{\partial t} \psi(x,t) = (i\bar{\phi} - m)\psi(x,t) + \eta(x,t), \quad \frac{\partial}{\partial t} \bar{\psi}(x,t) = \bar{\psi}(x,t)(-i\bar{\phi} - m) + \bar{\eta}(x,t)
  \]

- introduce retarded Green functions
  \[
  (\partial_t - i\bar{\phi}_x + m)G(x,t) = \delta^n(x)\delta(t) \quad \Rightarrow \quad G(x,t) = \theta(t) \int \frac{d^n p}{(2\pi)^n} e^{-(\bar{\phi}+m)t} e^{ipx}
  \]
  \[
  \tilde{G}(x,t)(\partial_t + i\bar{\phi}_x + m) = \delta^n(x)\delta(t) \quad \Rightarrow \quad \tilde{G}(x,t) = \theta(t) \int \frac{d^n p}{(2\pi)^n} e^{-(\bar{\phi}+m)t} e^{-ipx}
  \]
  - then, solutions are
    \[
    \psi(x,t) = \int d^n x' \frac{d^n p}{(2\pi)^n} e^{-(\bar{\phi}+m)t} e^{ip(x-x')} \psi(x',0) + \int_0^t dt' \int d^n x' \frac{d^n p}{(2\pi)^n} e^{-(\bar{\phi}+m)(t-t')} e^{ip(x-x')} \eta(x',t')
    \]
    \[
    \bar{\psi}(x,t) = \int d^n x' \frac{d^n p}{(2\pi)^n} \bar{\psi}(x',0) e^{-(\bar{\phi}+m)t} e^{-ip(x-x')} + \int_0^t dt' \int d^n x' \frac{d^n p}{(2\pi)^n} \bar{\eta}(x',t') e^{-(\bar{\phi}+m)(t-t')} e^{-ip(x-x')}
    \]

  * observe that large t behavior \( \sim e^{-(\bar{\phi}+m)t} \)
  - eigenvalues of \( \bar{\phi} + m \): \( \lambda_i = \pm i\sqrt{p^2 + m}, \quad (i = 1, \ldots, 4) \) ... doubly degenerated!
  - \( i\bar{\phi} \) is a hermitian matrix \( \Rightarrow \) diagonalized by a unitary transformation \( U^\dagger(p) = U^{-1}(p) \)
  - Green function behaves like
    \[
    G(x,t) = \theta(t) \int \frac{d^n p}{(2\pi)^n} U(p) \begin{pmatrix}
      e^{-(i\sqrt{p^2+m})t} & e^{-(i\sqrt{p^2+m})t} \\
      e^{-(i\sqrt{p^2+m})t} & e^{-(i\sqrt{p^2+m})t}
    \end{pmatrix} U^{-1}(p)e^{ipx}
    \]
    \[\rightarrow \sim e^{-mt} \quad \text{as} \quad t \rightarrow \infty \quad : \text{no damping factor for chiral fermion (m = 0)!} \quad \text{... not completely satisfactory} \]
6.2 Introduction of Kernel

- Langevin equation with a kernel ... bosonic case

\[
\dot{\phi}(x,t) = -\int d^n y K(x,y) \frac{\delta S}{\delta \phi(y,t)} + \eta(x,t) \quad \text{with} \quad \langle \eta(x,t) \rangle = 0, \quad \langle \eta(x,t)\eta(x',t') \rangle = 2K(x,x')\delta(t-t')
\]

- equivalent Fokker–Planck equation

\[
\dot{P}(\phi,t) = \int d^n x \frac{\delta}{\phi(x)} K(x,y) \left( \frac{\delta}{\delta \phi(y)} + \frac{\delta S}{\delta \phi(y)} \right) P(\phi,t)
\]

★ thus, a positive definite kernel \(K > 0\)

ensures the same stationary (equilibrium) distribution \(e^{-S}\), while relaxation process toward it is affected

★ Fermionic case

- role of kernel is slightly different

... crucial role in fermionic case, for it assures the convergence to equilibrium

- choose kernel

\[
K(x,y) = (i\partial_x + m)\delta^n(x-y)
\]

- introduce fermionic noises \(\theta_\alpha\) and \(\bar{\theta}_\alpha\) with statistical properties

\[
\langle \theta_\alpha \rangle = \langle \bar{\theta}_\alpha \rangle = 0, \quad \langle \theta_\alpha \theta_\beta \rangle = \langle \bar{\theta}_\alpha \bar{\theta}_\beta \rangle = 0, \quad \langle \theta_\alpha(x,t)\bar{\theta}_\beta(x',t') \rangle = 2(i\partial_x + m)_{\alpha\beta}\delta^n(x-x')\delta(t-t')
\]

- set up the Langevin equation (free case)

\[
\frac{\partial}{\partial t} \psi(x,t) = -\int d^n y K(x,y) \frac{\delta S}{\delta \psi(y,t)} + \theta(x,t) = (i\partial_x + m)(i\partial_x - m)\psi(x,t) + \theta(x,t)
\]

\[
= (\partial^2_x - m^2)\psi(x,t) + \theta(x,t)
\]

\[
\frac{\partial}{\partial t} \bar{\psi}(x,t) = \int d^n y \frac{\delta S}{\delta \bar{\psi}(y,t)} K(y,x) + \bar{\theta}(x,t) = \bar{\psi}(x,t)(-i\bar{\partial}_x - m)(-i\bar{\partial}_x + m) + \bar{\theta}(x,t)
\]

\[
= \bar{\psi}(x,t)(\bar{\partial}^2_x - m^2) + \bar{\theta}(x,t)
\]
- average over Grassmann noises

\[
\left\langle F(\theta, \bar{\theta}) \right\rangle = \mathcal{N} \int \mathcal{D}\theta \mathcal{D}\bar{\theta} F(\theta, \bar{\theta}) e^{-\frac{1}{2} \int d^n x dt \bar{\theta}(x, t)(i\bar{\phi} + m)^{-1}\theta(x, t)}
\]

\[
\int d\theta_a = \int d\bar{\theta}_a = 0, \quad \int d\theta_a \theta_a = \int d\bar{\theta}_a \bar{\theta}_a = \text{const.} \quad (a \equiv \{\alpha, x, t\})
\]

... show that this distribution reproduces the desired statistical properties of \(\theta\) and \(\bar{\theta}\) : home work

- interacting case

\[
\frac{\partial}{\partial t} \psi(x, t) = -\int d^n y K(x, y) \frac{\delta S}{\delta \psi(y, t)} + \theta(x, t) = (\partial_x^2 - m^2)\psi(x, t) - (i\partial_x + m) \frac{\delta S_{\text{int.}}}{\delta \psi(x, t)} + \theta(x, t)
\]

\[
\frac{\partial}{\partial t} \bar{\psi}(x, t) = \int d^n y \frac{\delta S}{\delta \bar{\psi}(y, t)} K(y, x) + \bar{\theta}(x, t) = \bar{\psi}(x, t) (\partial_x^2 - m^2) + \frac{\delta S_{\text{int.}}}{\delta \bar{\psi}(x, t)} (-i\bar{\phi}_x + m) + \bar{\theta}(x, t)
\]

solutions with initial conditions \(\psi(x, 0) = 0\) and \(\bar{\psi}(x, 0) = 0\)

\[
\psi(x, t) = \int_0^\infty dt' d^n x' G(x - x', t - t') \left\{ \theta(x', t') - (i\bar{\phi}_{x'} + m) \frac{\delta S_{\text{int.}}}{\delta \psi(x', t')} \right\}
\]

\[
\bar{\psi}(x, t) = \int_0^\infty dt' d^n x' \left\{ \bar{\theta}(x', t') + \frac{\delta S_{\text{int.}}}{\delta \bar{\psi}(x', t')} (-i\bar{\phi}_{x'} + m) \right\} G(x' - x, t - t')
\]

where \(G\) is the bosonic Green function

\[
G(x, t) = \theta(t) \int \frac{d^n p}{(2\pi)^n} e^{-(p^2 + m^2)t + ipx}
\]

- free propagator (2-point correlation function at the lowest order \(D\))
\[ \langle \psi_\alpha(x, t) \bar{\psi}_\beta(x', t') \rangle^{(0)} = \int_0^t dt_1 d^n x_1 \int_0^{t'} dt_2 d^n x_2 G(x - x_1, t - t_1) G(x_2 - x', t' - t_2) 2(i \bar{\theta}_x + m)_{\alpha \beta} \delta^n(x_1 - x_2) \delta(t_1 - t_2) \]
\[
= \int \frac{d^n p}{(2\pi)^n} \frac{(-p + m)_{\alpha \beta}}{p^2 + m^2} \left( e^{-(p^2 + m^2)|t - t'|} - e^{-(p^2 + m^2)(t + t')} \right) e^{ip(x - x')} \\
= \int \frac{d^n p}{(2\pi)^n} \frac{1}{p + m} \left( e^{-(p^2 + m^2)|t - t'|} - e^{-(p^2 + m^2)(t + t')} \right) e^{ip(x - x')} \\
\rightarrow \int \frac{d^n p}{(2\pi)^n} \frac{1}{p + m} \left( \frac{1}{p + m} \right)_{\alpha \beta} e^{ip(x - x')} \text{ as } t = t' \to \infty
\]

\[ 
\downarrow \text{ordinary Feynman propagator for fermion realized ... with the same damping factor as in the scalar case} \\
- \text{for ‘aged’ system (initial condition at } t = -\infty) \\
D_{\alpha \beta}(p, t) = \frac{(-p + m)_{\alpha \beta}}{p^2 + m^2} e^{-(p^2 + m^2)|t|} \Rightarrow \frac{\partial}{\partial t} D_{\alpha \beta}(p, t) = (-\bar{\theta} + m)_{\alpha \beta} (G(p, -t) - G(p, t))
\]

\[ * \text{ Stochastic quantization of fermion fields based on the Langevin equation with the above kernel gives just the same result as in the ordinary case, which could be shown order by order in perturbation (my personal guess)} \]

6.3 Fermionic Fokker–Planck Equation

\[ * \text{ probability interpretation of Fokker–Planck distribution with Grassmann fields??} \]
\[ \ldots \text{ formally ok(?)} \]

- Define Fokker–Planck distribution functional \( P[\psi, \bar{\psi}, t] \) as

\[ \forall F[\psi, \bar{\psi}], \quad \langle F[\psi, \bar{\psi}] \rangle_t = \mathcal{N} \int D\theta D\bar{\theta} \int F[\psi_t(\theta, \bar{\theta}), \bar{\psi}_t(\theta, \bar{\theta})] e^{-\frac{1}{2} \int d^n x d t \bar{\psi}(x, t) \left[ i \bar{\theta}_x + m \right]^{-1} \psi(x, t)} = \int D\psi D\bar{\psi} \int F[\psi, \bar{\psi}] P[\psi, \bar{\psi}, t] \\
\text{clearly} \\
\quad P[\psi, \bar{\psi}, t] = \mathcal{N} \int D\theta D\bar{\theta} \delta[\psi - \psi_t(\theta, \bar{\theta})] \delta[\bar{\psi} - \bar{\psi}_t(\theta, \bar{\theta})] e^{-\frac{1}{2} \int d^n x d t \bar{\psi}(x, t) \left[ i \bar{\theta}_x + m \right]^{-1} \psi(x, t)} \]
Derivation of Fokker–Planck equation (on the basis of Ito calculus)

Since

\[ F[\psi + d\psi, \bar{\psi} + d\bar{\psi}] = F[\psi, \bar{\psi} + d\bar{\psi}] + \int d^n x d\psi(x,t) \frac{\delta}{\delta \psi(x,t)} \left( F[\psi, \bar{\psi}] + \int d^n y d\bar{\psi}(y,t) \frac{\delta}{\delta \bar{\psi}(y,t)} F[\psi, \bar{\psi}] + O(\sqrt{dt}^3) \right) \]

\[ = F[\psi, \bar{\psi}] + \int d^n x \left( d\psi(x,t) \frac{\delta}{\delta \psi(x,t)} F[\psi, \bar{\psi}] + d\bar{\psi}(x,t) \frac{\delta}{\delta \bar{\psi}(x,t)} F[\psi, \bar{\psi}] \right) + \int d^n x d^n y \left( d\psi(x,t) \frac{\delta}{\delta \psi(x,t)} \right) \left( d\bar{\psi}(y,t) \frac{\delta}{\delta \bar{\psi}(y,t)} \right) F[\psi, \bar{\psi}] + O(\sqrt{dt}^3) \]

we have

\[ \langle dF[\psi, \bar{\psi}] \rangle = \left\langle \int d^n x \left( - \int d^n y K(x, y) \frac{\delta S}{\delta \psi(y,t)} dt + d\theta(x,t) \right) \frac{\delta}{\alpha \delta \psi\alpha(x,t)} F[\psi, \bar{\psi}] \right\rangle \]

\[ + \left\langle \int d^n x \left( \int d^n y \frac{\delta S}{\delta \psi(y,t)} K(y, x) dt + d\theta(x,t) \right) \frac{\delta}{\alpha \delta \psi\alpha(x,t)} F[\psi, \bar{\psi}] \right\rangle \]

\[ - \left\langle \int d^n x d^n y 2K_{\alpha\beta}(x,y) dt \frac{\delta}{\delta \psi\alpha(x,t)} \frac{\delta}{\delta \bar{\psi}\beta(y,t)} F[\psi, \bar{\psi}] \right\rangle \]

thus, Fokker–Planck equation reads as

\[ \frac{\partial}{\partial t} P[\psi, \bar{\psi}, t] = \int d^n x d^n y \left[ - \frac{\delta}{\delta \psi\alpha(x)} K_{\alpha\beta}(x,y) \frac{\delta S}{\delta \psi\beta(y)} + \frac{\delta}{\delta \psi\alpha(x)} K_{\beta\alpha}(y, x) \frac{\delta S}{\delta \psi\beta(y)} + 2 \frac{\delta}{\delta \psi\alpha(x)} K_{\alpha\beta}(x,y) \frac{\delta S}{\delta \psi\beta(y)} \right] P[\psi, \bar{\psi}, t] \]

... derive this result, paying attention to possible sign changes in Grassmann version of Leibniz rule: home work

- Transform \( P \) by \( e^{-\frac{1}{2}S} \)

\[ P[\psi, \bar{\psi}, t] = e^{-\frac{1}{2}S[\psi, \bar{\psi}]} \Psi[\psi, \bar{\psi}, t] \quad \Rightarrow \quad e^{\frac{1}{2}S} \frac{\delta}{\delta \psi} e^{-\frac{1}{2}S} = \frac{\delta}{\delta \psi} - \frac{1}{2} \frac{\delta S}{\delta \psi}, \quad e^{\frac{1}{2}S} \frac{\delta}{\delta \bar{\psi}} e^{-\frac{1}{2}S} = \frac{\delta}{\delta \bar{\psi}} - \frac{1}{2} \frac{\delta S}{\delta \bar{\psi}} \]
and
\[
\frac{\partial}{\partial t} \Psi[\psi, \bar{\psi}, t] = \int d^m x d^n y \left[ \left( \frac{\delta}{\delta \psi_\beta(y)} - \frac{1}{2} \frac{\delta S}{\delta \psi_\beta(y)} \right) K_{\alpha\beta}(x, y) \left( \frac{\delta}{\delta \psi_\alpha(x)} + \frac{1}{2} \frac{\delta S}{\delta \psi_\alpha(x)} \right) \\
- \left( \frac{\delta}{\delta \psi_\alpha(x)} - \frac{1}{2} \frac{\delta S}{\delta \psi_\alpha(x)} \right) K_{\alpha\beta}(x, y) \left( \frac{\delta}{\delta \psi_\beta(y)} + \frac{1}{2} \frac{\delta S}{\delta \psi_\beta(y)} \right) \right] \Psi[\psi, \bar{\psi}, t] \equiv -\hat{H} \Psi[\psi, \bar{\psi}, t]
\]

- (transformed) Fokker–Planck ‘Hamiltonian’
\[
\hat{H} = \int d^m x d^n y \left[ \left( -\frac{\delta}{\delta \psi_\beta(y)} + \frac{1}{2} \frac{\delta S}{\delta \psi_\beta(y)} \right) K_{\alpha\beta}(x, y) \left( \frac{\delta}{\delta \psi_\alpha(x)} + \frac{1}{2} \frac{\delta S}{\delta \psi_\alpha(x)} \right) \\
- \left( -\frac{\delta}{\delta \psi_\alpha(x)} + \frac{1}{2} \frac{\delta S}{\delta \psi_\alpha(x)} \right) K_{\alpha\beta}(x, y) \left( \frac{\delta}{\delta \psi_\beta(y)} + \frac{1}{2} \frac{\delta S}{\delta \psi_\beta(y)} \right) \right]
\]

* The eigenstate belonging to zero eigenvalue is easily found
\[
\Psi_0 \propto e^{-\frac{1}{2}S} \quad \text{... should be the ground state (for there is no node in } \Psi_0)
\]

- if eigenstates of \( \hat{H}, \hat{H} \Psi_n = E_n \Psi_n \), form a complete set, they can expand \( \Psi \) as
\[
\Psi[\psi, \bar{\psi}, t] = \sum_{n=0}^{\infty} a_n \Psi_n[\psi, \bar{\psi}] e^{-E_n t}
\]

○ if \( \Re(E_n) > 0, \forall n > 0 \), then the equilibrium distribution is nothing but
\[
P[\psi, \bar{\psi}, t] = e^{-\frac{1}{2}S} \Psi[\psi, \bar{\psi}, t] \xrightarrow{t \to \infty} e^{-\frac{1}{2}S} \Psi_0[\psi, \bar{\psi}, t] = \mathcal{N} e^{-S} \quad \text{with } \mathcal{N}^{-1} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S}
\]

⇒ ordinary field theory is then recovered!

Remarks

▷ The spectrum of \( \hat{H} \) is non-trivial ⇔ positivity is easily shown (at least formally) in bosonic cases with positive kernels
- maybe, perturbative proof possible
... nonperturbatively...??

... open question
free case ... spectrum can be explored explicitly

- Fokker–Planck Hamiltonian (before similarity transformation) reads as

\[
H = \int d^nx d^ny \left[ \frac{\delta}{\delta \bar{\psi}_\alpha(x)} K_{\alpha\beta}(x, y) \frac{\delta S}{\delta \psi_\beta(y)} - \frac{\delta}{\delta \psi_\alpha(x)} K_{\beta\alpha}(y, x) \frac{\delta S}{\delta \bar{\psi}_\beta(y)} - 2 \frac{\delta}{\delta \bar{\psi}_\beta(y)} K_{\alpha\beta}(x, y) \frac{\delta}{\delta \psi_\alpha(x)} \right]
\]

\[
= \int d^n \bar{x} \left[ \frac{\delta}{\delta \bar{\psi}_\alpha(x)} (-\partial_x^2 + m^2) \psi_\alpha(x) + \frac{\delta}{\delta \bar{\psi}_\alpha(x)} (\bar{\psi}_\alpha(x)(-\partial_x^2 + m^2)) + 2 \frac{\delta}{\delta \psi_\alpha(x)} (i \bar{\phi}_x + m) \frac{\delta}{\delta \bar{\psi}_\beta(x)} \right]
\]

- similarity transformation by

\[
V = \exp \left\{ \int d^n x \frac{\delta}{\delta \bar{\psi}_\alpha(x)} (-i \bar{\phi}_x + m)^{-1} \frac{\delta}{\delta \psi_\beta(x)} \right\}
\]

transforms

\[
V \psi_\alpha(x) V^{-1} = \psi_\alpha(x) - (-i \partial_x + m)^{-1} \frac{\delta}{\delta \psi_\beta(x)}, \quad V \bar{\psi}_\beta(x) V^{-1} = \bar{\psi}_\beta(x) + \frac{\delta}{\delta \psi_\alpha(x)} (i \bar{\phi}_x + m)^{-1} \frac{\delta}{\delta \bar{\psi}_\beta(x)}
\]

and therefore the spectrum of Hamiltonian is non-negative!

\[
VHV^{-1} = \int d^n \bar{x} \left[ \frac{\delta}{\delta \bar{\psi}_\alpha(x)} (-\partial_x^2 + m^2) \psi_\alpha(x) + \frac{\delta}{\delta \bar{\psi}_\alpha(x)} (\bar{\psi}_\alpha(x)(-\partial_x^2 + m^2)) \right]
\]

\[
= \int d^n \bar{x} \left[ \frac{\delta}{\delta \psi_\alpha(x)} (-\partial_x^2 + m^2) \psi_\alpha(x) + \frac{\delta}{\delta \psi_\alpha(x)} (-\partial_x^2 + m^2) \bar{\psi}_\alpha(x) \right]
\]

\[
= \int d^n \bar{x} \left[ \mathcal{A}^\dagger_\alpha(x)(-\partial_x^2 + m^2) \mathcal{A}_\alpha(x) + \mathcal{B}^\dagger_\alpha(x)(-\partial_x^2 + m^2) \mathcal{B}_\alpha(x) \right] \geq 0
\]

where the ‘coherent state’ representation has been introduced

\[
\mathcal{A} \leftrightarrow \psi, \quad \mathcal{A}^\dagger \leftrightarrow \frac{\delta}{\delta \psi}, \quad \mathcal{B} \leftrightarrow \bar{\psi}, \quad \mathcal{B}^\dagger \leftrightarrow \frac{\delta}{\delta \bar{\psi}}
\]

\[
\{\mathcal{A}, \mathcal{A}^\dagger\} = \{\mathcal{B}, \mathcal{B}^\dagger\} = 1 : \text{only nonvanishing anti-commutation relations}
\]

... confirm the above similarity transformation : homework
Chapter 7. STOCHASTIC QUANTIZATION IN MINKOWSKI SPACE

★ Possibility of obtaining Minkowski space Green functions without resort to Wick rotation from the Euclidean counterparts?

- Euclidean space and stochastic quantization : closely connected
  stochastic process $\leftrightarrow$ $\exists$ positive semi-definite probability $\leftrightarrow$ Euclidean space
- On the other hand, the Langevin equation is based on the classical equation of motion
  ... less restrictions (actually, in a sense, Lagrangian or Hamiltonian not required)

- The main problem :

Can one obtain the Feynman measure $e^{iS}$ as an equilibrium distribution?
With which Langevin or Fokker–Planck equation one has to start?

Scalar field

Naive expectation : Minkowski action $iS \Leftrightarrow -S_E$ : Euclidean action

$$\text{Action for a real scalar field : } S = \int d^nx \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}, \quad g_{\mu\nu} = \text{diag.}(1,-1,-1,-1)$$

$$\Leftrightarrow S_E = \int d^nx \left\{ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right\}$$

(Naive) Langevin equation in Minkowski space

$$\dot{\phi}(x,t) = i \frac{\delta S}{\delta \phi(x,t)} + \eta(x,t) \quad \Leftrightarrow \quad \dot{\phi} = -\frac{\delta S}{\delta \phi} + \eta$$

with a Gaussian white noise

$$\langle \eta(x,t) \rangle = 0, \quad \langle \eta(x,t) \eta(x',t') \rangle = 2\delta^n(x-x')\delta(t-t')$$

**Observe** : real field $\phi(x)$ unavoidably becomes complex-valued $\phi(x,t) \neq \phi^*(x,t)$ owing to the presence of $i$ in the drift term!
- no need to consider real $\eta$, complex $\eta$ can do as well, as long as the same statistical properties are maintained
solution of the above Langevin equation in free case $\lambda = 0$ (initial condition : $\phi(x,0) = 0$)

$$\phi(k,t) = \int_0^t dt' e^{i(k^2-m^2)(t-t')} \eta(k,t'), \quad \langle \eta(k,t)\eta(k',t') \rangle = 2(2\pi)^n \delta^n(k+k') \delta(t-t')$$

2-point correlation function (at the lowest order)

$$\langle \phi(k,t)\phi(k',t') \rangle = (2\pi)^n \delta^n(k+k') \frac{i}{k^2-m^2}(e^{i(k^2-m^2)|t-t'|} - e^{i(k^2-m^2)(t+t')})$$

\[ \xrightarrow{t=t'} (2\pi)^n \delta^n(k+k') \frac{i}{k^2-m^2}(1 - e^{2i(k^2-m^2)t}) \]

\[ \diamond \text{ No strict } t \rightarrow \infty \text{ limit exists, instead, } t \rightarrow \infty \text{ can be interpreted in the distribution sense!} \]

- relations (P = principal value)

$$\frac{1 - e^{ixt}}{x} = P \frac{1 - e^{ixt}}{x}, \quad \lim_{t \rightarrow \infty} e^{ixt} = 0, \quad \lim_{t \rightarrow \infty} P \frac{e^{ixt}}{x} = i\pi \delta(x)$$

... confirm these relations : home work

imply

$$\lim_{t=t' \rightarrow \infty} \langle \phi(k,t)\phi(k',t') \rangle = i(2\pi)^n \delta^n(k+k') \left(P \frac{1}{k^2-m^2} - i\pi(k^2-m^2)\right) = (2\pi)^n \delta^n(k+k') \frac{i}{k^2-m^2 + i\epsilon}$$

i.e., the ordinary Feynman propagator (with $i\epsilon$-prescription) for scalar field!

\[ \triangleright \text{ Notice no consistency of this interpretation has been checked to higher orders} \]

- less sophisticated trial : adding an (infinitesimal) imaginary mass $k^2 - m^2 \rightarrow k^2 - m^2 + i\epsilon$

... perturbatively, it would be clear that stochastic diagrams reproduce Feynman diagram in Minkowski space

(thanks to the damping factor $\sim e^{-\epsilon t}$)

\[ \star \text{ Fokker–Planck equation??} \]

- remember $\phi(x,t)$, subject to the Langevin equation

$$\dot{\phi} = i \frac{\delta S}{\delta \phi} + \eta$$
is complex-valued

\[ \phi(x,t) = \phi_r(x,t) + i\phi_i(x,t) \]

- real and imaginary parts of \( \phi \) satisfy (assume real-valued noise \( \eta \) and \( i\epsilon \) prescription)

\[ \dot{\phi}_r(x,t) = -\epsilon \phi_r(x,t) - (\partial^2 - m^2)\phi_i(x,t) + \eta(x,t), \quad \dot{\phi}_i(x,t) = (\partial^2 - m^2)\phi_r(x,t) - \epsilon \phi_i(x,t) \]

- Fokker–Planck equation reads as

\[
\frac{\partial}{\partial t} P[\phi_r, \phi_i] = \int d^n x \left\{ \frac{\delta}{\delta \phi_r(x)} \left( \frac{\delta}{\delta \phi_r(x)} + \epsilon \phi_r(x) + (\partial^2 - m^2)\phi_i(x) \right) + \frac{\delta}{\delta \phi_i(x)} \left( \epsilon \phi_i(x) - (\partial^2 - m^2)\phi_r(x) \right) \right\} P[\phi_r, \phi_i] \]

▷ exact solution is known \( \text{Ref. H.N. & Y. Yamanaka, PRD34(’86)492} \)

- damping factor \( \sim e^{-\epsilon t} \) allows us to consider its equilibrium limit

\[
P_{\text{eq}}[\phi_r, \phi_i] = \lim_{t \to \infty} P[\phi_r, \phi_i, t] = \exp \left\{ -\epsilon \int \frac{d^n k}{(2\pi)^n} \left( |\phi_r(k)|^2 + \frac{(k^2 - m^2)^2}{(k^2 - m^2)^4} |\phi_i(k)|^2 - \frac{2\epsilon}{k^2 - m^2} \phi_r(k)\phi_i(-k) \right) \right\}
\]

... confirm that it indeed is the stationary solution of the Fokker–Planck equation : home work

* a positive distribution functional of \( \phi_r \) and \( \phi_i \) ... probability interpretation ok! (as long as \( \epsilon > 0 \))

* it reproduces the standard Feynman propagator!

\[
\langle \phi(k,t)\phi(k',t) \rangle = \int \mathcal{D}\phi_r \mathcal{D}\phi_i (\phi_r(k,t) + i\phi_i(k,t))(\phi_r(k',t) + i\phi_i(k',t)) P[\phi_r, \phi_i, t]
\]

\[
= \langle (\phi_r(k,t)\phi_r(k',t) - \phi_i(k,t)\phi_i(k',t)) \rangle + i\langle (\phi_r(k,t)\phi_i(k',t) + \phi_i(k,t)\phi_r(k',t)) \rangle
\]

\[
\rightarrow (2\pi)^n \delta^n(k+k') \left( \frac{\epsilon}{(k^2 - m^2)^2 + \epsilon^2} + i \frac{k^2 - m^2}{(k^2 - m^2)^2 + \epsilon^2} \right) = (2\pi)^n \delta^n(k+k') \frac{i}{k^2 - m^2 + i\epsilon}
\]

... show this result : home work
Gauge field

- Naive Langevin equation in Minkowski space for gauge field may be

\[ \dot{A}_\mu(x,t) = i \frac{\delta S}{\delta A^\mu(x,t)} + \eta_\mu(x,t) \]

classical action

\[ S = -\frac{1}{4} \int d^n x F_{\mu\nu} F^{\mu\nu} \text{ : Abelian case} \quad \text{or} \quad S = -\frac{1}{2} \int d^n x \text{tr}(F_{\mu\nu} F^{\mu\nu}) \text{ : non-Abelian case} \]

Gaussian white noise

\[ \langle \eta_\mu(x,t) \rangle = 0, \quad \langle \eta_\mu(x,t) \eta_\nu(x',t) \rangle = -2g_{\mu\nu} \delta^n(x-x')\delta(t-t') \]

... metric tensor \(-g_{\mu\nu}\) is not positive definite!! ⇒ impossible to be realized as a probability

★ This reminds us of the apparent \textit{incompatibility} between manifest gauge invariance and Lorentz covariance (→ difficult to solve??)

No resolution has been proposed so far, as far as I know... ... open question
* Hidden supersymmetry in stochastic quantization – scalar field case –

- Recall, in functional approach, an “effective Fokker–Planck action” has appeared

\[ S_{FP} = \int dt d^nx \left\{ \frac{1}{4\kappa} \dot{\phi}^2 + \frac{\kappa}{4} \left( \frac{\delta S}{\delta \phi} \right)^2 - \bar{\psi} \left( \partial_{\tau} + \kappa \frac{\delta^2 S}{\delta \phi^2} \right) \psi \right\} \]

... Jacobian determinant \( \delta \eta / \delta \phi \) has been exponentiated in terms of fermion fields \( \psi \) and \( \bar{\psi} \)

- introduce an auxiliary boson field \( F \), by adding \( -\kappa (F - \frac{1}{2} \frac{\delta S}{\delta \phi})^2 \),

\[ S_{FP} = \int dt d^nx \left\{ \frac{1}{4\kappa} \dot{\phi}^2 - \kappa F^2 + \kappa F \frac{\delta S}{\delta \phi} - \bar{\psi} \left( \partial_{\tau} + \kappa \frac{\delta^2 S}{\delta \phi^2} \right) \psi \right\} \]

- Now, superfield formulation :

\[ \Phi \equiv \phi + \bar{\xi} \psi + \bar{\psi} \xi + \bar{\xi} \xi F, \quad D \equiv \frac{\partial}{\partial \xi} - \frac{\xi}{2\kappa} \frac{\partial}{\partial \tau}, \quad \bar{D} \equiv - \frac{\partial}{\partial \bar{\xi}} + \bar{\xi} \frac{\partial}{\partial \bar{\tau}} \]

\[ S_{FP} = \int dt d^nx d\xi d\bar{\xi} (\kappa \Phi \bar{D} D \Phi + \kappa \mathcal{L}(\Phi)) \]

*******

- actually

\[ \bar{D} D = - \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{\xi}} + \frac{1}{2\kappa} \frac{\partial}{\partial \tau} - \frac{\xi}{2\kappa} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \tau} + \frac{\bar{\xi}}{2\kappa} \frac{\partial}{\partial \bar{\xi}} \frac{\partial}{\partial \bar{\tau}} - \frac{\xi \bar{\xi}}{2(\kappa)^2} \frac{\partial^2}{\partial \tau^2} \]

\[ \bar{D} D \Phi = -F + \frac{1}{2\kappa} \frac{\partial \Phi}{\partial \tau} + \frac{\bar{\xi}}{\kappa} \frac{\partial \bar{\psi}}{\partial \tau} + \xi \bar{\xi} \left( \frac{1}{2\kappa} \frac{\partial \Phi}{\partial \xi} - \frac{1}{(2\kappa)^2} \frac{\partial^2 \Phi}{\partial \bar{\xi}^2} \right) \]

resulting in

\[ \Phi \bar{D} D \Phi \bigg|_{\xi \bar{\xi}} = \bar{\xi} \xi \left( \frac{1}{2\kappa} \frac{\partial}{\partial \tau} (\phi F - \frac{1}{2\kappa} \phi \dot{\phi}) + \frac{1}{(2\kappa)^2} \dot{\phi}^2 - F^2 - \frac{1}{\kappa} \bar{\psi} \psi \right) \]
\[ L(\Phi) \bigg|_{\xi} = \xi \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} L(\Phi) = \xi (F L'(\phi) - \bar{\psi} L''(\phi)) \]

- Superfield \( \Phi \) : assumed to be scalar under super translation (with appropriate parameters \( a \) and \( b \), to be determined later)

\[
\delta \tau = -\frac{1}{2\kappa} (a \bar{\xi} \epsilon + b \xi \bar{\epsilon}), \quad \delta \xi = \epsilon, \quad \delta \bar{\xi} = \bar{\epsilon}
\]

that is,

\[
0 = \Phi'(\tau', \xi', \bar{\xi}') - \Phi(\tau, \xi, \bar{\xi})
\]

\[
= \delta \tau \partial_\tau \Phi + \delta \xi \partial_\xi \Phi + \delta \bar{\xi} \partial_{\bar{\xi}} \Phi + \delta \phi + \bar{\xi} \delta \psi + \delta \bar{\psi} \xi + \bar{\xi} \delta F
\]

\[
= -\frac{1}{2\kappa} (a \bar{\xi} \epsilon + b \xi \bar{\epsilon})(\phi + \bar{\xi} \psi + \bar{\psi} \xi) + \epsilon (-\bar{\psi} - \bar{\xi} F) + \bar{\epsilon}(\psi + \xi F) + \delta \phi + \bar{\xi} \delta \psi + \delta \bar{\psi} \xi + \bar{\xi} \delta F
\]

implying

\[
\delta \phi = -(\bar{\epsilon} \psi + \bar{\psi} \epsilon), \quad \delta \psi = \epsilon \left( \frac{a}{2\kappa} \dot{\phi} - F \right), \quad \delta \bar{\psi} = \bar{\epsilon} \left( \frac{b}{2\kappa} \dot{\phi} - F \right), \quad \delta F = -\frac{1}{2\kappa} (a \bar{\psi} \epsilon + b \bar{\epsilon} \psi)
\]

Choose \( a = -b = 1 \), i.e.,

\[
\delta \phi = -(\bar{\epsilon} \psi + \bar{\psi} \epsilon), \quad \delta \psi = \epsilon \left( \frac{1}{2\kappa} \dot{\phi} - F \right), \quad \delta \bar{\psi} = -\bar{\epsilon} \left( \frac{1}{2\kappa} \dot{\phi} + F \right), \quad \delta F = -\frac{1}{2\kappa}(\bar{\psi} \epsilon - \bar{\epsilon} \psi)
\]

then, we have

\[
\delta \left\{ \frac{1}{4\kappa} \dot{\phi}^2 - \kappa F^2 + \kappa F \frac{\delta S}{\delta \phi} - \bar{\psi} \left( \partial_\tau + \kappa \frac{\delta^2 S}{\delta \phi^2} \right) \psi \right\} = \frac{\partial}{\partial \tau} \left( \bar{\psi} \epsilon (F - \frac{1}{2\kappa} \dot{\phi}) - \frac{1}{2} (\bar{\psi} \epsilon - \bar{\epsilon} \psi) \frac{\delta S}{\delta \phi} \right)
\]

\[ \Rightarrow \text{action is invariant under the super-transformation!} \]

- *Speculation* :
  if boundary conditions

\[
\bar{\psi}(0) = \bar{\psi}(-\infty) = 0, \quad F(0) = \frac{1}{2\kappa} \dot{\phi}(0)
\]
are allowed to be imposed, we may conclude

\[ 0 = \delta \{ \kappa (\Phi \bar{D}D\Phi + \mathcal{L}(\Phi)) \} = -(\delta \tau \partial_\tau + \delta \xi \partial_\xi + \delta \bar{\xi} \partial_{\bar{\xi}}) \{ \kappa (\Phi \bar{D}D\Phi + \mathcal{L}(\Phi)) \} \]

In particular, put \( \bar{\epsilon} = 0 \), then

\[ \delta \tau = -\frac{1}{2\kappa} \bar{\xi} \epsilon, \quad \delta \xi = \epsilon, \quad \delta \bar{\xi} = 0 \]

the function depends \( \tau, \xi, \bar{\xi} \), only through their invariant combination \( \tau + \frac{1}{2\kappa} \bar{\xi} \xi \)

\[ \kappa (\Phi \bar{D}D\Phi + \mathcal{L}(\Phi)) \equiv \mathcal{F}(\tau, \xi, \bar{\xi}) = \mathcal{F} \left( \tau + \frac{1}{2\kappa} \bar{\xi} \xi, 0, 0 \right) = \mathcal{F}(\tau, 0, 0) + \frac{1}{2\kappa} \bar{\xi} \xi \partial_\tau \mathcal{F}(\tau, 0, 0), \]

from which we may conclude

\[ S_{FP} = \frac{1}{2} S \]

this is just what we need to show!

******

---

* field variations corresponding to these ones should be, and actually are, consistent with the above boundary conditions
Interesting subjects, not treated here, include

- Stochastic process and supersymmetry
  - Parisi–Sourlas reduction supersymmetry, Nicolai map
- Large $N$ limit
- Stochastic regularizations
  - new regularization schemes ← new degree of freedom $t$
- Numerical applications
- Recent developments